

# High-Energy Astrophysics Notes

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# Chapter 1

## Astrophysical Fluid Dynamics

### 1.1 Fluid equations

By the conservation of mass; if the density of the fluid is  $\rho$ , the rate of change of the mass is

$$\frac{\partial}{\partial t} \int_V \rho dV \quad (1.1.1)$$

In the absence of sources or sinks for matter, this must be equal to the net inflow of mass integrated over the whole surface.

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho dV &= - \int_V \nabla \cdot (\rho \vec{u}) dV \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) &= 0 \quad \text{Eulerian} \end{aligned} \quad (1.1.2)$$

To write in lagrangian:

$$\begin{aligned} \frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho = -\nabla \cdot (\rho \vec{u}) + \vec{u} \cdot \nabla \rho = -\rho \nabla \cdot \vec{u} \\ \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} &= 0 \quad \text{Lagrangian} \end{aligned} \quad (1.1.3)$$

The force per unit area is

$$\vec{F} = p \hat{s} \quad (1.1.4)$$

where  $p$  is the pressure and  $\hat{s}$  is the unit normal to the surface.

Using the stress tensor

$$\vec{F} = \sigma_{ij} \hat{s} \quad (1.1.5)$$

where  $\sigma_{ij}$  are the components of the stress tensor.

The diagonal components are the pressure components  $p\delta_{ij}$  and the rest is the residual tensor.

The force is

$$F = - \int_S p \hat{n} \cdot d\vec{S} = - \int_V \nabla \cdot (p\hat{n}) dV$$

The rate of change of momentum is

$$\left( \frac{D}{Dt} \int_V \rho \vec{u} dV \right) \cdot \hat{n} = - \int_V \nabla \cdot (p\hat{n}) dV + \int_V \rho \vec{g} \cdot \hat{n} dV$$

where the second term is the gravitational force ( $mg$ ) on the element.

$$\nabla \cdot (p\hat{n}) = \hat{n} \cdot \nabla p + p \nabla \cdot \hat{n}$$

and the second term of the right is zero because  $\hat{n}$  is a unit vector in a constant direction. Then, in the limit when the fluid lump is small, we can replace  $\int dV$  by  $\delta V$ , and then we have

$$\begin{aligned} \frac{D}{Dt} (\rho \vec{u} \delta V) \cdot \hat{n} &= -\delta V \hat{n} \cdot \nabla p + \delta V \rho \vec{g} \cdot \hat{n} \\ \hat{n} \vec{u} \frac{D}{Dt} (\rho \delta V) + \rho \delta V \hat{n} \cdot \frac{D \vec{u}}{Dt} &= -\delta V \hat{n} \cdot \nabla p + \delta V \rho \vec{g} \cdot \hat{n} \\ \rho \delta V \hat{n} \cdot \frac{D \vec{u}}{Dt} &= -\delta V \hat{n} \cdot \nabla p + \delta V \rho \vec{g} \cdot \hat{n} \end{aligned}$$

This is true for all  $\delta V$  and  $\hat{n}$ , and so finally we have

$$\rho \frac{D \vec{u}}{Dt} = -\nabla p + \rho \vec{g} \quad \text{Lagrangian} \quad (1.1.6)$$

This is the Lagrangian form of the momentum equation: the momentum of a fluid element changes as a result of pressure gradients and gravitational forces. To transform to the Eulerian form we use the relation

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla p + \rho \vec{g} \quad (1.1.7)$$

This says that the momentum contained in a fixed grid cell within the fluid changes as a result of pressure and gravitational forces, and any imbalance in the momentum flux in and out of the grid cell.

## 1.2 Gravitation

Any force that is conservative (i.e. one in which the work done is a closed loop  $\oint \vec{F} \cdot d\vec{l} = 0$ ) can be written as

$$\vec{F} = \nabla \Phi \quad (1.2.1)$$

where  $\Phi$  is the scalar potential. We can see that this is the case by using Stokes' theorem, i.e.  $\oint \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$ . Since  $\oint \vec{F} \cdot d\vec{l} = 0$  for conservative force fields then  $\nabla \times \vec{F}$  must be zero. Since 'curl of grad' is always zero, any force field of this form is conservative.

In the case of gravity we define a (scalar) gravitational potential  $\Psi$  such that the gravitational acceleration  $\vec{g}$  is given by

$$\vec{g} = -\nabla\Psi \quad (1.2.2)$$

The work required to take a unit mass to infinity is

$$\int_r^\infty \vec{g} \cdot d\vec{l} = \int_r^\infty \nabla\Psi \cdot d\vec{l} = \Psi_\infty - \Psi(r) \quad (1.2.3)$$

consider  $\Psi = -\frac{GM}{r}$ , then

$$\vec{g} = -\nabla\Psi = -\frac{GM}{r^2}\hat{r}$$

for a point mass located at  $\vec{r}'$ , the potential is

$$\Psi = -\frac{GM}{|\vec{r} - \vec{r}'|},$$

and, for a system of points

$$\Psi = -\sum_i \frac{GM_i}{|\vec{r} - \vec{r}'_i|} \quad (1.2.4)$$

Now, if we have  $\vec{g} \cdot d\vec{S}$ , then

$$\int_S \vec{g} \cdot d\vec{S} = -\int_S \frac{GM}{r^2}\hat{r} \cdot d\vec{S} = -GM \int_S d\Omega = -4\pi GM,$$

so

$$\int_S \vec{g} \cdot d\vec{S} = -4\pi GM \quad (1.2.5)$$

If we distribute masses throughout the volume enclosed by S, then

$$\int_S \vec{g} \cdot d\vec{S} = -4\pi G \sum_i M_i = -4\pi G \int_V \rho dV$$

$$\int_S \vec{g} \cdot d\vec{S} = \int_V \nabla \cdot \vec{g} dV,$$

and so

$$\int_V (\nabla \cdot \vec{g} + 4\pi G\rho) dV = 0$$

or

$$\nabla^2\Psi = 4\pi G\rho \quad (1.2.6)$$

### 1.2.1 The virial theorem

Examining the second derivative of the moment of inertia of the particle about the origin we have

$$\begin{aligned} \frac{d^2(mr^2)}{dt^2} &= 2m \frac{d}{dt} \left( \vec{r} \cdot \frac{d\vec{r}}{dt} \right) \\ &= 2m \vec{r} \cdot \frac{d^2\vec{r}}{dt^2} + 2m \left( \frac{d\vec{r}}{dt} \right)^2 \\ &= 2m \left( \frac{d\vec{r}}{dt} \right)^2 + 2\vec{r} \cdot \vec{F} \end{aligned}$$

The quantity  $\vec{r} \cdot \vec{F}$  is defined as the virial  $V$  and  $T$  as the total kinetic energy. Hence, summing this over all particles we have

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2T + \sum (\vec{r} \cdot \vec{F}) \quad (1.2.7)$$

We can define  $V$  for two particles acting on each other with  $\vec{F}_{i(j)} = -\vec{F}_{j(i)}$  and the contribution of this pair of forces to the virial is then  $\vec{F}_{i(j)} \cdot (\vec{r}_i - \vec{r}_j)$ , so

$$V = \sum_i \sum_{j>i} \vec{F}_{i(j)} \cdot (\vec{r}_i - \vec{r}_j) \quad (1.2.8)$$

where the summation is over all pair of particles and  $j>i$  to avoid double counting. If the ideal gas laws apply we have collisional processes with  $\vec{r}_i - \vec{r}_j = 0$ , then all forces except gravitational forces may be neglected and we have

$$V = - \sum_i \sum_{j>i} \frac{Gm_i m_j}{r_{ij}}$$

where  $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

The each term is simply the work done in separating the pair of particles to infinity against gravitational attraction, so  $V$  is the potential energy,  $\Omega$  of the cloud of particles under consideration.

If the system is in a steady state,  $I$  is constant, and consequently we have

$$2T + \Omega = 0 \quad (1.2.9)$$

and, for fluids, dividing the total kinetic energy into kinetic energy of the mean flow locally and the additional kinetic energy stored in particle motions in the rest frame of the local fluid element that is the thermal energy of the fluid  $U$ , so

$$2T_k + 2U + \Omega = 0 \quad (1.2.10)$$

where  $T_k$  is the thermal kinetic energy contained in the mean streaming motion of the particles at every point.

### 1.3 The energy equation

Now we have two important eulerian equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \quad (1.3.1)$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla p - \rho \nabla \Psi. \quad (1.3.2)$$

#### 1.3.1 Barotropic equations of state: the isothermal and adiabatic cases

For an ideal gas the equation of state is given by

$$p = \frac{\Re}{\mu} \rho T \quad (1.3.3)$$

where  $\Re$  is the gas constant

$$\bar{d}Q = d\varepsilon + pdV \quad (1.3.4)$$

where  $\bar{d}Q$  is the quantity of heat absorbed by unit mass of fluid from surroundings,  $pdV$  is the work done by unit mass of fluid if its volume changes by  $dV$  and  $d\varepsilon$  is the change in the internal energy content of unit mass of the fluid.

Conventionally the changes in heat absorbed and work done are expressed using the Pfaffian operator  $\bar{d}$ , which denote that the change in these quantities in going from an initial state to a final one depends on the route taken through thermodynamic phase space.

If  $\varepsilon = \varepsilon(T)$ , so the energy equation can be written

$$\bar{d}Q = \frac{d\varepsilon}{dT} dT + pdV \quad (1.3.5)$$

alternatively, in the form

$$\bar{d}Q = C_V dT + \frac{\Re T}{\mu V} dV \quad (1.3.6)$$

where the specific heat capacity at constant volume is defined to be

$$C_V = \frac{d\varepsilon}{dT} \quad (1.3.7)$$

In the case of a gas undergoing a reversible adiabatic change we have

$$\begin{aligned} C_V dT \frac{\Re T}{\mu V} &= 0 \\ \Rightarrow C_V d \ln T + \frac{\Re}{\mu} d \ln V &= 0 \end{aligned}$$

$$\Rightarrow V \sim T^{-\frac{C_V}{\mathfrak{R}/\mu}} \quad (1.3.8)$$

substituting for the ideal gas equation of state, we also obtain

$$p \sim T^{1+\frac{C_V}{\mathfrak{R}/\mu}} \quad (1.3.9)$$

and

$$p \sim V^{-1+\frac{\mathfrak{R}}{\mu C_V}} \quad (1.3.10)$$

Using the specific heat capacity at constant pressure

$$pdV + Vdp = \frac{\mathfrak{R}}{\mu} dT \quad (1.3.11)$$

Thus, from (3.5) we have

$$dQ = \left( \frac{d\varepsilon}{dT} + \frac{\mathfrak{R}}{\mu} \right) dT - Vdp \quad (1.3.12)$$

from which it follows that

$$C_p = \frac{d\varepsilon}{dT} + \frac{\mathfrak{R}}{\mu} \quad (1.3.13)$$

and

$$C_p - C_V = \frac{\mathfrak{R}}{\mu} \quad (1.3.14)$$

and we define  $\gamma$  as the ratio of specific heats, i.e.

$$\gamma = \frac{C_p}{C_V} \quad (1.3.15)$$

Thus the relations between  $p$ ,  $V$  and  $T$  for reversible adiabatic changes may be written

$$V \sim T^{-1/(\gamma-1)} \quad (1.3.16)$$

$$p \sim T^{\gamma/(\gamma-1)} \quad (1.3.17)$$

$$p \sim T^{-\gamma} \quad (1.3.18)$$

Bearing in mind that the volume in the above equations refers to that occupied by unit mass of gas, the density is just the reciprocal of  $V$  and thus we can write the equation of state of gas undergoing reversible adiabatic changes in the barotropic form:

$$p = K\rho^\gamma \quad (1.3.19)$$

where  $K$  is constant.

Returning now to the first law of thermodynamics (3.4) we note that for a reversible change this can be written  $dQ = TdS$ , where  $S$  is the entropy per unit mass.

### 1.3.2 Energy equation

We know from the first law of thermodynamics that in the absence of viscous (dissipative) processes, the internal energy in a fluid,  $\varepsilon$  per kg, is related to the  $pdV$  work done,  $W$ , and the energy gained from the surroundings,  $Q$ , by

$$\frac{D\varepsilon}{Dt} = \frac{dW}{dt} + \frac{dQ}{dt}, \quad (1.3.20)$$

the variation of  $W$  does not depend on pressure

$$\frac{DW}{Dt} = -p \frac{D(1/\rho)}{Dt} = \frac{p}{\rho^2} \frac{D\rho}{Dt} \quad (1.3.21)$$

so

$$\frac{D\varepsilon}{Dt} = \frac{p}{\rho^2} \frac{D\rho}{Dt} - \dot{Q}_{cool} \quad (1.3.22)$$

Here we have defined a cooling  $\dot{Q}_{cool}$  (per kilogram) which is positive if the medium is cooled and negative for external heating.

We now define a total energy per unit volume

$$E = \rho \left( \frac{1}{2} u^2 + \Psi + \varepsilon \right) \quad (1.3.23)$$

that implies

$$\frac{DE}{Dt} = \frac{E}{\rho} \frac{D\rho}{Dt} + \rho \left( \vec{u} \cdot \frac{D\vec{u}}{Dt} + \frac{D\Psi}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt} - \dot{Q}_{cool} \right). \quad (1.3.24)$$

Now we substitute for  $\frac{D\rho}{Dt}$  and  $\frac{D\vec{u}}{Dt}$  from the continuity and momentum equations, and convert from Lagrangian to Eulerian form:

$$\frac{DE}{Dt} = \frac{\partial E}{\partial t} + \vec{u} \cdot \nabla E \quad (1.3.25)$$

For the RHS of equations (3.24) and (3.25) we use the Lagrangian forms of the continuity and momentum equations, so

$$\frac{E}{\rho} \frac{D\rho}{Dt} = -\frac{E}{\rho} \rho \nabla \cdot \vec{u} \quad (1.3.26)$$

$$\rho \vec{u} \cdot \frac{D\vec{u}}{Dt} = \vec{u} \cdot (-\nabla p - \rho \nabla \Psi) \quad (1.3.27)$$

$$\rho \frac{D\Psi}{Dt} = \rho \frac{\partial \Psi}{\partial t} + \rho \vec{u} \cdot \nabla \Psi \quad (1.3.28)$$

$$\rho \frac{p}{\rho^2} \frac{D\rho}{Dt} = -\rho \frac{p}{\rho^2} \rho \nabla \cdot \vec{u} = -p \nabla \cdot \vec{u} \quad (1.3.29)$$

so

$$\frac{DE}{Dt} = -E\nabla \cdot \vec{u} - \vec{u} \cdot \nabla p - \rho \vec{u} \cdot \nabla \Psi + \rho \frac{\partial \Psi}{\partial t} + \rho \vec{u} \cdot \nabla \Psi - p \nabla \cdot \vec{u} - \rho \dot{Q}_{cool} \quad (1.3.30)$$

$$= -(E + p)\nabla \cdot \vec{u} - \vec{u} \cdot \nabla p + \rho \frac{\partial \Psi}{\partial t} - \rho \dot{Q}_{cool} \quad (1.3.31)$$

Putting this together with (42)

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\vec{u}] = -\rho \dot{Q}_{cool} + \rho \frac{\partial \Psi}{\partial t} \quad (1.3.32)$$

For almost every situation we wish to contemplate  $\frac{\partial \Psi}{\partial t} = 0$ , i.e.  $\Psi$  is a function of position only. then the only things we have to do are relate  $E$  to the quantities we have been dealing with ( $\rho$ ,  $T$ , etc.) and sort out an expression for  $\dot{Q}_{cool}$ .

Concerning  $\dot{Q}_{cool}$ , we have another new variable which we have to determine, for each element in the fluid. So we have apparently merely shifted the problem of closure of the set of equations again, but this time it is more straightforward to see what to do about it. Heating and cooling rates are something we can appreciate (and work out) in an astrophysical context more easily.

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<sup>1</sup>[1]

# Bibliography

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