Tensor Notes

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January 29, 2021

Contents

Tensors are important in many areas of science and I, as a physicist and a astrophysical researcher, have tried to put the basics of it at these very brief "Tensor Notes".

I hope you enjoy!

Chapter 1

Introduction

1.1 The index notation

An easy way to introduce the index notation is using a matrix as it follows

$$\begin{pmatrix} v_1^{'} \\ \vdots \\ v_n^{'} \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \tag{1.1.1}$$

and, as we can see, the prime terms can be written like

$$v_n^{'} = \sum_{m=1}^{n} A_{nm} v_m$$

A similar case is the inner product,

$$\overrightarrow{v} \cdot \overrightarrow{u} = \sum_{m=1}^{n} v_m u_m$$

The inner product case here is a simple one, but soon we will go further into the inner products.

It is useful to omit the summation symbols in index notation,

$$\sum_{m=1}^{n} A_{nm} v_m \implies A_{nm} v_m$$

$$\sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} A_{\alpha\beta} B_{\beta\gamma} C_{\gamma} \implies A_{\alpha\beta} B_{\beta\gamma} C_{\gamma}$$

Note that a summation is assumed over the indices that appear twice and no summation is assumed when indices appear only once.

1.2 Invariant, covariant and contravariant vectors

We can explain the basics of these two definitions using the concept of base. We have an orthogonal base and an unknown vector, if this base scales changes from meters to centimeters (divided by 100) we have three options to this vector change:

- The vector does not change, because it does not depend on the unit scale, so it is an invariant one;
- The vector changes proportionally, because it is proportional to the unit scale, so it is a covariant one;
- The vector changes inversely, because it is inversely proportional to the unit scale, so it is a contravariant one.

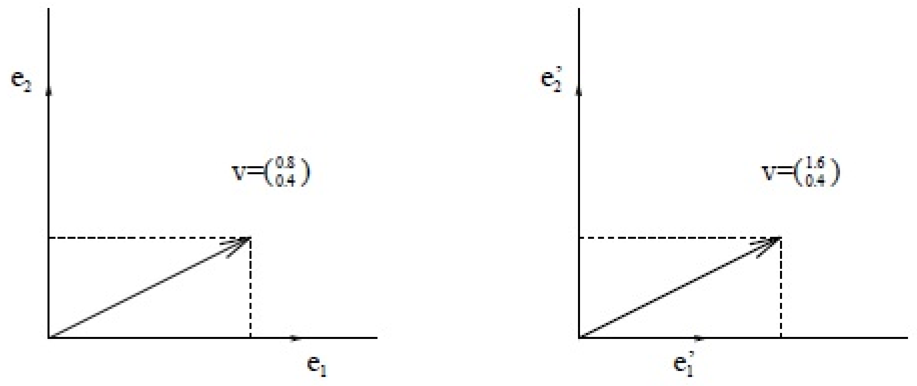
If we construct a basis transformation of a contravariant vector like,

$$\overrightarrow{e}'_1 = a_{11}\overrightarrow{e}_1 + a_{12}\overrightarrow{e}_2,$$
$$\overrightarrow{e}'_2 = a_{21}\overrightarrow{e}_1 + a_{22}\overrightarrow{e}_2$$

that we can also describe like,

$$\begin{pmatrix}\overrightarrow{e}'_1\\ \overrightarrow{e}'_2\end{pmatrix} = \begin{pmatrix} projection\ of\ \overrightarrow{e}'_1\ onto\ \overrightarrow{e}_1 & projection\ of\ \overrightarrow{e}'_1\ onto\ \overrightarrow{e}_2 \\ projection\ of\ \overrightarrow{e}'_2\ onto\ \overrightarrow{e}_1 & projection\ of\ \overrightarrow{e}'_2\ onto\ \overrightarrow{e}_2\end{pmatrix}\begin{pmatrix}\overrightarrow{e}_1\\ \overrightarrow{e}_2\end{pmatrix} \tag{1.2.1}$$

so, as we can see in the image 2.1 that follows, the vector transformation can



be described like,

$$\begin{pmatrix}\overrightarrow{v}'_1\\ \overrightarrow{v}'_2\end{pmatrix} = \begin{pmatrix} projection\ of\ \overrightarrow{e}_1\ onto\ \overrightarrow{e}'_1 & projection\ of\ \overrightarrow{e}_2\ onto\ \overrightarrow{e}'_1 \\ projection\ of\ \overrightarrow{e}_1\ onto\ \overrightarrow{e}'_2 & projection\ of\ \overrightarrow{e}_2\ onto\ \overrightarrow{e}'_2\end{pmatrix}\begin{pmatrix}\overrightarrow{v}_1\\ \overrightarrow{v}_2\end{pmatrix} \tag{1.2.2}$$

So, we have to replace the primed elements by nom-primed elements and vise-versa, but also have to transpose the matrix. So, we can express this transformation by,

$$\overrightarrow{v}' = (\Lambda^{-1})^T \overrightarrow{v} \tag{1.2.3}$$

Showing these results Mathematically: We define a gradient of a function by,

$$(\nabla f)_\mu \equiv \frac{\partial f}{\partial x_\mu} \equiv w_\mu \tag{1.2.4}$$

and a vector fild defined in this manifold $V : \overrightarrow{v} = \overrightarrow{v}(\overrightarrow{x}')$. So we perform a homogeneous linear transformation of the coordinates:

$$x^{'}_\mu = A_{\mu\nu} x_\nu. \tag{1.2.5}$$

So, in this case not only the coordinates x_μ change (and therefore the dependence of $\overrightarrow{v}$ on the coordinates), but also the components of the vectors,

$$v^{'}_\mu \overrightarrow{x} = A_{\mu\nu} v_\nu \overrightarrow{x}, \tag{1.2.6}$$

where the matrices are the same.

Now, using the chain rule,

$$\frac{\partial f}{\partial x^{'}_\mu} = \left(\frac{\partial x_\nu}{\partial x^{'}_\mu}\right) \frac{\partial f}{\partial x_\nu}. \tag{1.2.7}$$

Now, let us work with this expression into the parentheses using the equation (1.2.5),

$$x_\mu = (A^{-1})_{\mu\nu} x^{'}_\nu \tag{1.2.8}$$

$$\frac{\partial x_\mu}{\partial x^{'}_\alpha} = (A^{-1})_{\mu\nu} \frac{\partial x_\nu}{\partial x^{'}_\alpha} = (A^{-1})_{\mu\nu} \delta_{\nu\alpha} = (A^{-1})_{\mu\alpha} \tag{1.2.9}$$

so,

$$w^{'}_\mu = (A^{-1})^T_{\mu\nu} w_\nu \tag{1.2.10}$$

$$w^{'} = (A^{-1})^T w \tag{1.2.11}$$

Note that the matrix A denotes the coordinate transformation from coordinates x to the coordinates $x^{'}$.

Now, let us introduce some notations:

- We denote contravariant vectors with superscript indices, w^α;
- We denote covariant vectors with subscript indices, w_α;

- Matrix A can be written like $A: \ A^{\mu}{}_{\nu}$;
- The transposed version of A can be written like $A^T: \ A_{\nu}{}^{\mu}$;
- We denote a transformation like $v'^{\mu} = A^{\mu}{}_{\nu} v^{\nu}$;
- So $w'_{\mu} = (A^{-1})^{T}{}_{\mu}{}^{\nu} w_{\nu} = (A^{-1})^{\nu}{}_{\mu} w_{\nu}$;
- The delta have a matrix form like $\delta_{\mu\nu}: \ \delta^{\mu}{}_{\nu}$;
- So $\delta^{\mu}{}_{\nu} y^{\nu} = y^{\mu}$.

1.3 Introduction to tensors

1.3.1 Inner product and tensors

The inner product as we know,

$$s = \overrightarrow{a} \cdot \overrightarrow{b} = a^{\mu} b^{\mu} \tag{1.3.1}$$

does not have this property in general,

$$s' = \overrightarrow{a}' \cdot \overrightarrow{b}' = A^{\mu}{}_{\alpha} a^{\alpha} A^{\mu}{}_{\beta} b^{\beta} = (A^T)_{\alpha}{}^{\mu} A^{\mu}{}_{\beta} a^{\alpha} b^{\beta}, \tag{1.3.2}$$

where A is the transformation matrix. Only if A^{-1} equals A^T (i.e. if we are dealing with orthonormal transformations) s will not change. The matrices will then together form the kronecker delta $\delta_{\beta\alpha}$.

The inner product between a vector x and a covector y, however, is invariant under all transformations,

$$s = x^{\mu} y_{\mu}, \tag{1.3.3}$$

because for all A we can write

$$s' = x'^{\mu} y'_{\mu} = A^{\mu}{}_{\alpha} x^{\alpha} (A^{-1})^{\beta}{}_{\mu} y_{\beta} = (A^{-1})^{\beta}{}_{\mu} A^{\mu}{}_{\alpha} x^{\alpha} y_{\beta} = \delta^{\beta}{}_{\alpha} x^{\alpha} y_{\beta} = s \tag{1.3.4}$$

With help of this inner produce we can introduce a new inner product between two contravariant vectors which also has this invariance property. To do this, we introduce a covector w_{μ} and define the inner product between x^{μ} and y^{ν} with respect to this covector w_{μ} in the following way:

$$s = w_{\mu} w_{\nu} x^{\mu} y^{\nu}. \tag{1.3.5}$$

This inner product s will now obviously transform correctly, because it is made out of two invariant ones,

$$\begin{aligned} s^{'} &= (A^{-1})^{\mu}{}_{\alpha} w_{\mu} (A^{-1})^{\nu}{}_{\beta} w_{\nu} A^{\alpha}{}_{\rho} x^{\rho} A^{\beta}{}_{\sigma} y^{\sigma} \\ &= (A^{-1})^{\mu}{}_{\alpha} A^{\alpha}{}_{\rho} (A^{-1})^{\nu}{}_{\beta} A^{\beta}{}_{\sigma} w_{\mu} w_{\nu} x^{\rho} y^{\sigma} \\ &= \delta^{\mu}{}_{\rho} \delta^{\nu}{}_{\sigma} w_{\mu} w_{\nu} x^{\rho} y^{\sigma} \\ &= w_{\mu} w_{\nu} x^{\mu} y^{\nu} \\ &= s. \end{aligned} \tag{1.3.6}$$

We have now produced an invariant "inner product" for contravariant vectors by using a covariant vector w_μ as a measure of length.

Another form of represent this case is naming the $w_\mu w_\nu$ as $g_{\mu\nu}$,

$$s = g_{\mu\nu} x^{\mu} y^{\nu} \tag{1.3.7}$$

To produce the "old" inner product we just have to choose an orthonormal coordinate system,

$$s = g_{\mu\nu} x^{\mu} y^{\nu} = x^1 y^1 + x^2 y^2 + x^3 y^3, \tag{1.3.8}$$

so $g_{\mu\nu}$ is an identity.

The beauty of tensors is that they can have an arbitrary number of indices. One can also produce, for instance, a tensor with 3 indices,

$$A_{\alpha\beta\gamma} = x_{\alpha} y_{\beta} z_{\gamma}. \tag{1.3.9}$$

This is an ordered set of numbers labeled with three indices. It can be visualized as a kind of "super-matrix" in three dimensions.

These are tensors of rank 3, as opposed to tensors of rank 0 (scalars), rank 1 (vectors and covectors) and rank 2 (matrices and the other kind of tensors we introduced so far). We can distinguish between the contravariant rank and covariant rank. Clearly $A_{\alpha\beta\gamma}$ is a tensor of covariant rank 3 and contravariant rank 0. Its total rank is 3. We can also produce tensors of, for instance, contravariant rank 2 and covariant rank 3 (i.e. total rank 5): $B^{\alpha\beta}{}_{\mu\nu\phi}$, for example.

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[2] Wikipedia contributors. Covariance and contravariance of vectors — Wikipedia, the free encyclopedia.