Induced morphisms between Heyting-valued models

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Abstract

To the best of our knowledge, there are very few results on how Heyting-valued models are affected by the morphisms on the complete Heyting algebras that determine them: the only cases found in the literature are concerning automorphisms of complete Boolean algebras and complete embedding between them (*i.e.*, injective Boolean algebra homomorphisms that preserves arbitrary suprema and arbitrary infima). In the present work, we consider and explore how more general kinds of morphisms between complete Heyting algebras $\mathbb{H}$ and $\mathbb{H}'$ induce arrows between $\mathbf{V}^{(\mathbb{H})}$ and $\mathbf{V}^{(\mathbb{H}')}$, and between their corresponding localic toposes $\mathbf{Set}^{(\mathbb{H})}$ ($\simeq \mathbf{Sh}(\mathbb{H})$) and $\mathbf{Set}^{(\mathbb{H}')}$ ($\simeq \mathbf{Sh}(\mathbb{H}')$). In more detail: any *geometric morphism* $f^* : \mathbf{Set}^{(\mathbb{H})} \to \mathbf{Set}^{(\mathbb{H}')}$ (that automatically came from a unique locale morphism $f : \mathbb{H} \to \mathbb{H}'$), can be "lifted" to an arrow $\tilde{f} : \mathbf{V}^{(\mathbb{H})} \to \mathbf{V}^{(\mathbb{H}')}$. We also provide also some semantic preservation results concerning this arrow $\tilde{f} : \mathbf{V}^{(\mathbb{H})} \to \mathbf{V}^{(\mathbb{H}')}$.

1 Locales, sheaves and topos

1.1 Intuition

If $\langle X, \mathcal{O}(X)\rangle$ is a topological space, then the family of sets of real continuous functions has the property that given a open covering of an open set U, and a family of functions defined on elements of the covering that coincide on their pairwise intersections, there exists a unique "glueing" of that family into a function $U \to \mathbb{R}$. Sheaves attempt to capture the idea of objects locally defined giving rise to glueings.

Definition 1.1 *Let $\langle X, \mathcal{O}(X)\rangle$ be a topological space. Regard the poset $(\mathcal{O}(X), \subseteq)$ as a category, a presheaf on X is a functor $F : \mathcal{O}(X)^{op} \to \mathbf{Set}$. A sheaf on X is a presheaf F such that, for every open $U \in \mathcal{O}(X)$ and every open covering $\{U_i \in \mathcal{O}(X) \mid i \in I\}$ of U, the diagram below is an equalizer:*

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{\langle i,j\rangle \in I\times I} F(U_i \cap U_j)$$

We denote the category of presheaves on X by $\mathbf{Psh}(X)$ and the category of sheaves on X by $\mathbf{Sh}(X)$.

Notice that the definition of a sheaf depends only on the lattice of opens, therefore we may define presheaves and sheaves for any *locale* $\langle \mathbb{H}, \leq\rangle$, *i.e.* a complete lattice satisfying the following distributive law:

$$a \wedge \bigvee_{i\in I} c_i = \bigvee_{i \in I} a \wedge c_i.$$

Locales are precisely the complete Heyting algebras (cHA), where

$$a \to b = \bigvee\{c \in \mathbb{H} : a \wedge c \leq b\}.$$

It is also possible to define sheaves in more general categories, using Grothendieck topologies.

1.2 Development

Definition 1.2 *Let $\mathcal{C}$ be a small category. A Grothendieck topology on $\mathcal{C}$ is a function J which assigns to each object $c \in$ Obj$(\mathcal{C})$ a family $J(c)$ of sieves on c, satisfying some technical conditions. A pair $(\mathcal{C}, J)$ is called a (small) site.*

Every locale $(\mathbb{H}, \leq)$ gives rise to a Grothendieck topology: if $c \in \mathbb{H}$, then $J(c)$ is the set of all coverings of c that are downward closed. Another important example is the Zariski topology in Algebraic Geometry.

A Grothendieck topos is a category which is equivalent to the topos of sheaves on a site. Some properties of Grothendieck topos are interesting for developing logic in the context of category theory, as it contains a subobject classifier and is Cartesian closed.

Definition 1.3 *A topos is said to be localic if it is equivalent to the topos of sheaves on a locale.*

Theorem 1.4 *For a Grothendieck topos $\mathcal{T}$, the following conditions are equivalent:*

1. *$\mathcal{T}$ is a localic topos;*
2. *the subobjects of the terminal object constitute a family of generators of $\mathcal{T}$.*

A continuous function between topological spaces defines a $(\wedge, \bigvee)$-preserving morphism between the locales of open sets, and a geometric morphism between the corresponding sheaf topos: That is, (φ_*, φ^*) is a pair of functors such that $\varphi^* \dashv \varphi_*$ and φ^* preserves finite limits.

This mapping from the category of topological spaces to the category of topos and geometric morphisms is not full nor faithful. However, the mapping from the category of locales to the category of topos and geometric morphisms is fully faithful:

2 Heyting valued expansions of $\mathbf{V}$

Definition 2.1 *Locale-Valued Model*

We define, for a locale $\mathbb{H}$, the universe of $\mathbb{H}$-names by ordinal recursion. Given an ordinal α let

$$\mathbf{V}^{(\mathbb{H})}_\alpha = \left\{ f \in \mathbb{H}^X \mid \exists \beta < \alpha, X \subseteq \mathbf{V}^{(\mathbb{H})}_\beta \right\}$$

It is readily seen that $\mathbf{V}^{(\mathbb{H})}_\alpha \subset \mathbf{V}^{(\mathbb{H})}_{\alpha+1}$ and that for limit ordinals it is simply the union of the earlier stages. So we let the (proper class) $\mathbf{V}^{(\mathbb{H})}$ be defined as:

$$\mathbf{V}^{(\mathbb{H})} = \bigcup_{\alpha \in \mathrm{On}} \mathbf{V}^{(\mathbb{H})}_\alpha$$

Definition 2.2 *Atomic Formulas' Values*

We endow this class with two binary function on $\mathbb{H}$, namely $[\![\cdot \in \cdot]\!]$ and $[\![\cdot = \cdot]\!]$ defined by simultaneous recursion on

$$\langle x, y\rangle \prec \langle u, v\rangle \iff (x = u \wedge y \in \mathbf{dom}(v)) \vee (x \in \mathbf{dom}(u) \wedge y = v)$$

This is a well founded relation on $\mathbf{V}^{(\mathbb{H})} \times \mathbf{V}^{(\mathbb{H})}$. For now belief suffices and we shall define the two functions of which we spoke.

$$\begin{aligned}
[\![\cdot \in \cdot]\!] &: \mathbf{V}^{(\mathbb{H})} \times \mathbf{V}^{(\mathbb{H})} \to \mathbb{H}\\
&\langle x, y\rangle \to \bigvee_{u \in \mathbf{dom}(y)} (y(u) \wedge [\![x = u]\!])\\
[\![\cdot = \cdot]\!] &: \mathbf{V}^{(\mathbb{H})} \times \mathbf{V}^{(\mathbb{H})} \to \mathbb{H}\\
&\langle x, y\rangle \to \bigwedge_{\substack{u \in \mathbf{dom}(y)\\ v \in \mathbf{dom}(x)}} (y(u) \to [\![u \in x]\!]) \wedge (x(v) \to [\![v \in y]\!])
\end{aligned}$$

Definition 2.3 *Valuation of Complex Formulae*

The definition of valuation for atomic formulae can be extended to give values in $\mathbb{H}$ to any sentence in the language of ZF enriched with constant symbols for elements in $\mathbf{V}^{(\mathbb{H})}$. This is done, naturally, recursively: the binary connectives and negation correspond to the lattice's meet, join, implication *and pseudocomplementation; quantifications correspond to big meets ($\forall$) and big joins ($\exists$) over the entire $\mathbf{V}^{(\mathbb{H})}$.*

We, thus, write $[\![\phi(x_1 \cdots x_n)]\!]_{\mathbb{H}}$ for the value of the formula ϕ with its free variables substituted by the constants $x_1, \cdots, x_n \in \mathbf{V}^{(\mathbb{H})}$.

Theorem 2.4 *$\mathbf{V}^{(\mathbb{H})}$ is a Model of Intuitionistic ZF*

Model in the sense that $\mathbf{V}^{(\mathbb{H})}$ always values the axioms of Intuitionistic Logic and Set Theory (IZF) Theory as $1_{\mathbb{H}}$ and modus ponens *and other such intuitionistic inference rules are all valid for the semantic $\mathbf{V}^{(\mathbb{H})} \vDash \phi \iff [\![\phi]\!]_{\mathbb{H}} = 1_{\mathbb{H}}$. If $\mathbb{H}$ is a Boolean algebra, then $\mathbf{V}^{(\mathbb{H})}$ is, as above, a (classical) model of ZFC. And $\mathbf{V}^{(2)} \cong \mathbf{V}$ in a model theoretical sense.*

3 $\mathbf{V}^{(\mathbb{H})}$ and Descriptions of $\mathbf{Sh}(\mathbb{H})$

In this section we present, for the reader's convenience, an equivalent description of category of sheaves of a cHA $\mathbb{H}$, $\mathbf{Sh}(\mathbb{H}) \simeq \mathbb{H}$-$\mathbf{Set} \simeq \mathbf{Set}^{(\mathbb{H})}$, where the former equivalence is described in [Bor08c] and later is obtained by the cumulative hierarchy $\mathbf{V}^{(\mathbb{H})}$ by taking quotients as below:

Definition 3.1 *Consider the equivalence relation in $\mathbf{V}^{(\mathbb{H})}$ given by $x \equiv y$ if, and only if, $[\![x = y]\!] = 1$. The category $\mathbf{Set}^{(\mathbb{H})}$ is defined as:*

$$\mathit{Obj}\left(\mathbf{Set}^{(\mathbb{H})}\right) := {\mathbf{V}^{(\mathbb{H})}}/{\equiv}$$

$$\mathbf{Set}^{(\mathbb{H})}([x],[y]) := \left\{ [\phi] \in \mathbf{Set}^{(\mathbb{H})} \;\middle|\; [\![\mathit{fun}(\phi : x \to y)]\!] = 1 \right\}$$

The arrows do not depend on the choice of representative of the equivalence classes $[x]$ and $[y]$. The composition and identity are defined as in $\mathbf{Set}$.

4 Induced morphisms in Heyting valued models

There is an injection $\mathbf{V} \to \mathbf{V}^{(\mathbb{H})}$ given by $\hat{\cdot}$ which preserves the truth values of Σ_1 formulas, *i.e.*: $\psi(x_1 \cdots x_n) \iff [\![\psi(\hat{x}_1 \cdots \hat{x}_n)]\!] = 1_{\mathbb{H}}$. Currently, it is known that if $\phi : \mathbb{H} \to \mathbb{H}'$ is a complete and injective morfism of Heyting algebras, we can define a map $\tilde{\phi} : \mathbf{V}^{(\mathbb{H})} \to \mathbf{V}^{(\mathbb{H}')}$ that is injective and such that: for all $x, y \in \mathbf{V}^{(\mathbb{H})}$,

$$\phi\,[\![x = y]\!]_{\mathbb{H}} = \left[\!\left[\tilde{\phi}(x) = \tilde{\phi}(y)\right]\!\right]_{\mathbb{H}'}$$

$$\phi\,[\![x \in y]\!]_{\mathbb{H}} = \left[\!\left[\tilde{\phi}(x) \in \tilde{\phi}(y)\right]\!\right]_{\mathbb{H}'}$$

For Δ_0 formulas, the equality, trivially, still holds. One gets the following inequalities for any Σ_1 formula ψ:

$$\phi\,[\![\psi(x_1 \cdots x_n)]\!]_{\mathbb{H}} \leq \left[\!\left[\psi(\tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n))\right]\!\right]_{\mathbb{H}'}$$

4.1 Induced morphisms

Definition 4.1 *Generalized Connection between $\mathbf{V}^{(\mathbb{H})}$s*

Let $\phi : \mathbb{H} \to \mathbb{H}'$ function between complete Heyting algebras that preserving $(\bigvee, \wedge)$. Define the following compatible family of relations by ordinal recursion:

$$\begin{aligned}
x\,\tilde{\phi}_\alpha\,y \iff & \exists(\epsilon : \mathbf{dom}(x) \twoheadrightarrow \mathbf{dom}(y)) : (y \circ \epsilon = \phi \circ x) \wedge\\
& \forall u \in \mathbf{dom}(x) : \exists v \in \mathbf{V}^{(\mathbb{H}')} :\\
& \exists \beta < \alpha : (u\,\tilde{\phi}_\beta\,v) \wedge [\![v = \epsilon(u)]\!] = 1
\end{aligned}$$

$$\tilde{\phi} = \bigcup_{\alpha \in \mathrm{On}} \tilde{\phi}_\alpha$$

Proposition 4.2 *If ϕ is injective, then, for all $\alpha \in \mathrm{On}$, $\tilde{\phi}_\alpha$ is an injective function. In this case, the definition coincides with that found in [Bel05].*

Remark 4.3 *Naïve attempts to extend the definition found in [Bel05] are fated to fail, for in the absence of injectivity, relations defined without something similar to the $[\![v = \epsilon(u)]\!] = 1$ condition are not functions, and even as relations may have very limited (Small) domains.*

There are a handful of alternative definitions, which are all equivalent up to quotient by $\equiv$.

Remark 4.4 *We observe that $\mathbf{V}^{(\mathbb{H})}$ is a proper class (for $\mathbb{H} \neq \{0\}$), since there exists an injection $V \rightarrowtail \mathbf{V}^{(\mathbb{H})}$. Hence it can be shown that, for all $x \in \mathbf{V}^{(\mathbb{H})}$, $\left\{ y \in \mathbf{V}^{(\mathbb{H})} \mid [\![x = y]\!] = 1 \right\}$ is a proper class. Indeed, for all $\Sigma \subseteq \mathbf{V}^{(\mathbb{H})}$ such that $\Sigma \cap \mathbf{dom}(x) = \emptyset$, we may define $y_\Sigma : \mathbf{dom}(x) \cup \Sigma \to \mathbb{H}$ as:*

$$y_\Sigma(u) = \begin{cases} x(u) & \text{, if } u \in \mathbf{dom}(x)\\ 0 & \text{, if } u \in \Sigma \end{cases}$$

so that $[\![x = y_\Sigma]\!] = 1$.

Theorem 4.5 *If $\phi : \mathbb{H} \to \mathbb{H}'$, we have managed to show the domain of the relation $\tilde{\phi}$ to be the whole $\mathbf{V}^{(\mathbb{H})}$.*

4.2 Main results

Theorem 4.6 *For all $\langle x, x'\rangle, \langle y, y'\rangle, \langle z, z'\rangle \in \tilde{f}$,*

$$f([\![y \in x]\!]) \leq' [\![y' \in x']\!]' \quad \text{and} \quad f([\![x = z]\!]) \leq' [\![x' = z']\!]'$$

Corollary 4.7 *Let φ be a positive formula with bounded quantifiers and $f : \mathbb{H} \to \mathbb{H}'$. Then, for all $\langle a_1, a_1'\rangle, ..., \langle a_n, a_n'\rangle \in \tilde{f}$, we have:*

$$f\left([\![\varphi(a_1, ..., a_n)]\!]_{\mathbb{H}}\right) \leq [\![\varphi(a_1', ..., a_n')]\!]_{\mathbb{H}'}$$

Another consequence of the previous theorem is that, if $[\![x = z]\!]_{\mathbb{H}} = 1_{\mathbb{H}}$, then, since $1_{\mathbb{H}} = \bigwedge \emptyset$, we obtain:

$$f\,[\![x = z]\!]_{\mathbb{H}} = f(1_{\mathbb{H}}) = 1_{\mathbb{H}'} \leq [\![x' = z']\!]_{\mathbb{H}'}$$

that is, $[\![x' = z']\!]_{\mathbb{H}'} = 1_{\mathbb{H}'}$. Therefore, when we take the quotient by $\equiv$, the "semi-function" (i.e., a relation with total domain) $\tilde{f}$ defines an object mapping $\overline{f} : \mathrm{Obj}\left(\mathbf{Set}^{(\mathbb{H})}\right) \to \mathrm{Obj}\left(\mathbf{Set}^{(\mathbb{H}')}\right)$.

Proposition 4.8

1. *$\overline{id_{\mathbb{H}}} = id_{\mathbf{Set}^{(\mathbb{H})}} : \mathbf{Set}^{(\mathbb{H})} \to \mathbf{Set}^{(\mathbb{H})}$;*
2. *if $f' : \mathbb{H}' \to \mathbb{H}''$ preserves finite meets and arbitrary joins, then $\overline{f'} \circ \overline{f} = \overline{f' \circ f} : \mathbf{Set}^{(\mathbb{H})} \to \mathbf{Set}^{(\mathbb{H}'')}$.*

Proposition 4.9 *Let $\langle x, x'\rangle \in \tilde{f}$ with $\varepsilon : \mathbf{dom}(x) \twoheadrightarrow \mathbf{dom}(x')$ as witness. Consider the function $\varepsilon^{\mathbb{H}'} : \mathbf{dom}(x) \times \mathbf{dom}(x') \to \mathbb{H}'$ given by, for all $\langle u, v'\rangle \in \mathbf{dom}(x) \times \mathbf{dom}(x')$:*

$$\varepsilon^{\mathbb{H}'}(u, v') := f\,[\![u \in x]\!]_{\mathbb{H}} \wedge [\![\varepsilon(u) = v']\!]_{\mathbb{H}'} \wedge [\![v' \in x']\!]_{\mathbb{H}'}$$

Then, $\varepsilon^{\mathbb{H}'}$ determines a morphism of $\mathbb{H}$-sets (see e.g. [Bor08c]) $\varepsilon^{\mathbb{H}'} : (\mathbf{dom}(x), f \circ \delta_x) \to (\mathbf{dom}(x'), \delta_{x'})$ which does not depend on the choice of witness, where

$$\delta_x(u, v) := [\![u \in x]\!]_{\mathbb{H}} \wedge [\![u = v]\!]_{\mathbb{H}}, \text{ for all } u, v \in \mathbf{dom}(x)$$

$$\delta_{x'}(u', v') := [\![u' \in x']\!]_{\mathbb{H}'} \wedge [\![u' = v']\!]_{\mathbb{H}'}, \text{ for all } u', v' \in \mathbf{dom}(x')$$

Therefore, observing the proof of the aforementioned theorem, note that we may obtain these inequalities (and, thus, that $\varepsilon^{\mathbb{H}'}$ is iso) at least in the case that $f : \mathbb{H} \to \mathbb{H}'$ preserves (strictly) the implication and both arbitrary meets and joins. With that hypothesis, we could also adapt the corollary to the theorem to obtain the strict preservation of $\mathbb{H}$-values of all formulas with bounded quantifiers.

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