Categorical Properties of $\mathbb{Q}$-Sets

Enc$^{ue}_{o}$ntro de Lógica

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Quantales

$$(\mathbb{Q}, \odot, I, \leq, \wedge, \vee, \top, \bot)$$

$(\mathbb{Q}, \odot, I)$ is a monoid and $(\mathbb{Q}, \leq, \wedge, \vee, \top, \bot)$ is a complete lattice.

Distributivity law holds for $\odot$:

$$a \odot \bigvee_i b_i = \bigvee_i a \odot b_i \qquad \left(\bigvee_i b_i\right) \odot a = \bigvee_i b_i \odot a$$

it does not, in general, hold for $\wedge$; if that was the case, we would have a Heyting algebra with an additional monoid structure on it.

Types of Quantales

A quantale is said to be

1. *Commutative* whenever $\odot$ is;
2. *Idempotent* whenever $\odot$ is;
3. *Integral* or *Semicartesian* whenever the monoid's identity coincides with $\top$; in which case one also has $a \odot b \leq a \wedge b$.
4. *right(left)-sided* if $x \odot I \leq x$ ($I \odot x \leq x$);
5. (Right or Left) *Divisible*[3] when you have

$$\forall y : \forall x \leq y : \exists z : x = y \odot z$$

Remark: a semicartesian idempotent quantale is commutative and $\odot = \wedge$

[3] very useful property

Examples of Quantale

$$\mathbb{R}_L = (\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \leq^{\mathrm{op}}, \wedge = \max, \vee = \min, \top = 0, \bot = \infty)$$

$$\mathbb{N}_{\mathrm{div}} = (\mathbb{N}, \cdot, a \leq b \iff b \mid a, \mathrm{gcf}, \mathrm{lcm}, 0, 1)$$

Any complete MV/Heyting Algebra

Ideals of Commutative Rings with Unity

$\mathbb{R}_L$ is particularly important, as the categories enriched $\mathbb{R}_L$-Cat are the Lawvere metric spaces with contractions between them and its internal logic is a large part of "Continuous Logic" which extends Fuzzy ($[0, 1]$-logic) in a natural way.

From now on, all quantales are assumed to be semicartesian and commutative.

$\mathbb{Q}$-sets

A $\mathbb{Q}$-set is an ordinary set endowed with a $\mathbb{Q}$-equality $\delta : X \times X \to \mathbb{Q}$ satisfying the following axioms:

1. $$\delta(x, y) = \delta(y, x)$$
2. $$\delta(x, y) \odot \delta(y, z) \leq \delta(x, z)$$
3. $$\delta(x, y) \odot \delta(x, x) = \delta(x, y)$$

   This generalizes the notion of sheaves over a complete Heyting algebra introduced by Dana Scott.

Intuitively, δ measures objects' "overlappedness" and "sameness" graded on $\mathbb{Q}$; $\top$ being fully overlapped and $\perp$ being fully disjoint

Notation:

$$Ex := \delta(x, x)$$

called "extent", is how "not-foggy" or "solid" x is.

Extensional $\mathbb{Q}$-Sets

A Q-Set X is said to be extensional if any of these equivalent properties hold:

- $$(Ex = \delta(x, y) = Ey) \implies (x = y)$$
- $$\forall x, y \in X : (\forall z \in X : \delta(x, z) = \delta(y, z)) \implies (x = y)$$

Subsets and Singletons

We can generalize subsets and singletons of a $\mathbb{Q}$-set (X, δ) as maps

$$S : X \to \mathbb{Q}$$

Subset:

$$Sx \odot Ex = Sx$$

$$Sx \odot \delta(x, y) \leq Sy$$

Singleton: Subset $+$

$$Sx \odot Sy \leq \delta(x, y)$$

Provided X is extensional, we can define Singleton Equivalence:

$$S \sim S' \stackrel{\text{def}}{\iff} \forall x, y \in X : (Sx \odot Sy = Sx) \iff (S'x \odot S'y = S'x)$$

Compatibility and Gluing Condition

A subset $A \subseteq |X|$ is said *Compatible* if, and only if:

$$\forall a, b \in A : Ea \odot Eb = \delta(a, b)$$

We say that X satisfies the *Gluing Condition* if, and only if, for all compatible $A \subseteq |X|$ exists some x_A (called *a* "gluing of A") such that:

1. $$\forall a \in A : \delta(a, x_A) = Ea$$
2. $$\bigvee_{a \in A} Ea = Ex_A$$

A subset $M \subseteq$ is *maximal compatible* if, and only if, M is compatible and for all compatible $A \subseteq X$, if their gluing elements are the same (i.e. $x_A = x_M$), then $A \subseteq M$

Representable Singletons, Separable and Complete $\mathbb{Q}$-Sets

The *represented* singleton of an element $x \in X$ is: $S_x(y) = \delta(x, y)$

A $\mathbb{Q}$-Set X is *separable* iff. $x \mapsto S_x$ is injective.

For all separable $\mathbb{Q}$-Set X, its completion is:

$$SX = \{S_x : x \in X\}/\sim$$

And there is a canonical map:

$$\begin{aligned} Sng_X : X &\to SX \\ x &\mapsto [S_x] \end{aligned}$$

Analogously, a $\mathbb{Q}$-Set *complete* if, and only if, Sng_X is bijective.

Equivalent conditions

The following conditions are equivalent:

- X is extensional
- X is separable

Analogously, the following conditions are also equivalent:

- X satisfies the gluing condition
- X is complete

Relational Morphisms

The *Relational morphisms* conditions are "functional relation"-like conditions:

$$\varphi : X \times Y \to \mathbb{Q}$$

$$\varphi(x,y) \odot \delta'(y,y') \leq \varphi(x,y') \tag{1}$$
$$\delta(x,x') \odot \varphi(x,y) \leq \varphi(x',y) \tag{2}$$
$$\varphi(x,y) \odot \varphi(x,y') \leq \delta'(y,y') \tag{3}$$

$$Ex \odot \varphi(x,y) = \varphi(x,y) \tag{4}$$
$$\varphi(x,y) \odot Ey = \varphi(x,y) \tag{5}$$
$$\bigvee_{y \in Y} \varphi(x,y) = Ex \tag{6}$$

Functional Morphisms

Any function $f : X \to Y$ can be a *functional morphism* between (X, δ) and (Y, δ') provided it satisfies these two laws:

1. $$\delta(x, y) \leq \delta'(f(x), f(y))$$
2. $$Ef(x) = Ex$$

Manifold Categories

For any quantale $\mathbb{Q}$, there are a few possible categories of $\mathbb{Q}$-Sets, which is why we've made this handy chart:

	Relational	Functional
Just Bad	$\mathbb{Q}_{\mathsf{bad}r}$-Set	$\mathbb{Q}_{\mathsf{bad}f}$-Set
Separable	$\mathbb{Q}_{sr}$-Set	$\mathbb{Q}_{sf}$-Set
Complete	$\mathbb{Q}_{cr}$-Set	$\mathbb{Q}_{cf}$-Set

There are horizontal functors connecting them; in general there are "more" functional morphisms than relational ones, but one does have:

$$\mathbb{Q}_{cr}\text{-Set} \cong \mathbb{Q}_{cf}\text{-Set}$$

There are vertical functors (full subcategory inclusions) going down as well, but it isn't quite clear if they have adjoints;

$$\mathbb{Q}_{s\_}\text{-Set} \hookrightarrow \mathbb{Q}_{c\_}\text{-Set}$$

however, the above one seems to have an adjoint "completion".

Categorical Properties

All of these categories:

- Are complete and cocomplete;
- Have monoidal structure;
- Have subobject classifier for regular morphisms. (but we really don't have enough time to go into details)

For categories of the $\mathbb{Q}_{c_-}$-Set categories, all monos are regular.

Limits

Limits are similar to $\mathbb{H}$-Set – their Complete Heyting/Locale/Frame counterparts;

- The terminal object is the set of idempotents of $\mathbb{Q}$ with $\delta = \odot = \wedge$;
- Non-empty products are fibred by extent so that projections work and its δ is the infimum of the coordinates, for instance:

$$|X \times Y| = \{(x, y) \in |X| \times |Y| \ : \ Ex = Ey\}$$

- Equalizers are similar to Set's:

$$Eq(f, g) := \{x \in X \mid f(x) = g(x)\}$$

$$Eq(f, g) := \left\{ x \in X \ : \ \bigvee_{y \in Y} \varphi(x, y) = \psi(x, y) \right\}$$

The initial object in non-complete ($\mathbb{Q}_{s\_}$-Set and $\mathbb{Q}_{\mathrm{bad}\_}$-Set) categories is the empty set $\varnothing$. In complete categories, the initial object ends up being $\{\star\}$.

The colimits in non-complete $\mathbb{Q}$-Sets are the disjoint union with $\delta(x, y) = \delta_i(x, y)$ if both are in the same component X_i and $\delta(x, y) = \perp$ otherwise.

In complete $\mathbb{Q}$-Sets, we need to take the quotient of this disjoint union by an adequate equivalence relation.

Coequalizers are, again, similar to Set's. In the complete case, we need to do the same kind of quotient.

Monoidal Structure

One can define a monoid structure $\odot$, which is defined similarly to the regular cartesian product, but

$$\delta_{\odot}((x, x'), (y, y')) = \delta(x, x') \odot \delta(y, y')$$

as opposed to doing it with the infimum.

Subobject Classifier

$$\Omega_c = (\mathbb{Q} \times Idem(\mathbb{Q})/\sim, \delta_c)$$

$$(p, i) \sim (q, j) \Leftrightarrow i = j \leq ((p \to q) \wedge (q \to p))$$

$$\delta_c((p, i), (q, j)) = ((p \to q) \wedge (q \to p)) \odot i \odot j$$

For $f : A \to B$ monomorphism, $! : A \to 1$ is $!(a) = Ea$,
$\chi_f : B \to \Omega$, $\chi_f(b) = [(\bigvee_{a \in A} \delta(f(a), b)), Eb]$

Internal Logic

The internal logic of this category is similar to internal logic in Topos. Futhermore, we can define the $\odot$ as follows using the canonical arrow $can : \Omega \otimes \Omega \to \Omega \times \Omega$

$$\begin{tikzcd}
\Omega \otimes \Omega \arrow[r, "can"] \arrow[d, "!"'] & \Omega \times \Omega \arrow[d, "\odot"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

Closing Remarks

Our current goal is to understand that these categories' internal logic; which we suspect are related – in a as of yet imprecise way – to Linear/Affine Logic and Continuous Logic.

Our hope is that those categories and their properties meaningfully relate to the kind of quantale they are based on, which could potentially lead to applications in the areas where they arise more naturally: Algebra and Analysis.

Thank you