Alon-Boppana Bounds for Non-Regular Expanders

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Expansion Ratio and Expander Graphs

Let $G = (V, E)$ be a graph with n vertices. The **expansion ratio** of G is defined as

$$h(G) := \min\left\{\frac{|\delta(S)|}{\min\{|S|, |\overline{S}|\}} : \emptyset \neq S \subset V\right\}.$$

where $\delta(S)$ denotes the cut induced by S and $\overline{S} := V \setminus S$ is the complement of S. Let $(G_n)_{n\in\mathbb{N}}$ be a family of d-regular graphs, with $d \geq 2$, such that $\lim\limits_{n\to\infty} |V(G_n)| = \infty$. The family $(G_n)_{n\in\mathbb{N}}$ is called a **family of expanders** if there is $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$, one has $h(G_n) \geq \varepsilon$.

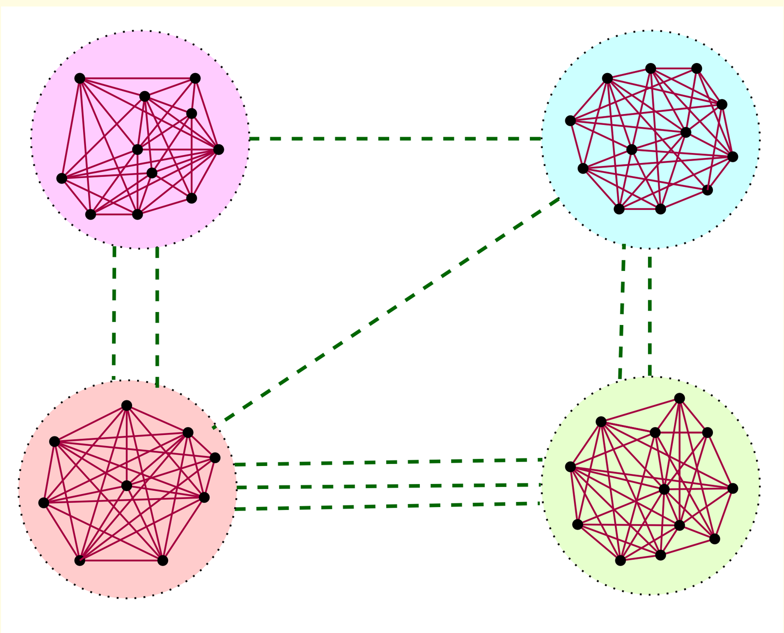
Conductance

The **conductance** of a nonempty set $S \subset V$ is

$$\phi_G(S) := \frac{w(\delta(S))}{\min\{\mathrm{vol}_G(S), \mathrm{vol}_G(\overline{S})\}}, \quad \text{where } \mathrm{vol}_G(S) := \sum_{v\in S} \deg_G(v).$$

The **conductance** of the graph$G$ is

$$\phi(G) := \min\left\{\phi_G(S) : \emptyset \neq S \subset V\right\}.$$



The notion of conductance is useful for identifying *clusters*. We call a set $S \subseteq V$ a *cluster* if the number of edges between vertices in S is considerably bigger than the size of $\delta(S)$. Expander graphs have high conductance, hence their vertices lie in a single cluster. In the image, the clusters are represented by the colored circles, so each vertex inside the same circle is in the same cluster. Each dotted line represents one edge between a vertex of one cluster to another vertex of the other cluster, i.e., edges between different colored circles. We can see that the number of edges crossing clusters is considerably smaller than the edges inside the clusters.

Cheeger's inequality and Alon-Boppana Bound

Cheeger's inequality says that, for every d-regular graph G,

$$\frac{d - \lambda_2^{\downarrow}(A_G)}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2^{\downarrow}(A_G))},$$

where $\lambda_2^{\downarrow}(A_G)$ denotes the second largest eigenvalue of the adjacency matrix of G. Recall that d is the largest eigenvalue of the adjacency matrix A_G of d-regular graphs. Hence, by Cheeger's inequality, one can use the difference $d - \lambda_2^{\downarrow}(A_G)$ of the two largest eigenvalues of A_G in place of $h(G)$ in the definition of expanders. Alon and Boppana bounded the second largest eigenvalue of A_G:

$$\lambda_2^{\downarrow}(A_G) \geq 2\sqrt{d-1}(1 - o(1)).$$

From the bound proved by Alon and Boppana it was defined a class of expanders that are cosidered optimal, which are called Ramanujan graphs.

Expander Graphs Properties

In expander graphs, the number of edges of any cut is approximately the number of expected edges on the respective cut of a random graph with the same expected number of edges. This property is known as Expander Mixing Lemma, which says that for every d-regular graph G with n vertices,

$$\left||E(S,T)| - \frac{d|S||T|}{n}\right| \leq \max\{|\lambda_2^{\downarrow}(A_G)|, |\lambda_{\min}(A_G)|\}\sqrt{|S||T|}, \quad \text{for all } S, T \subseteq V(G).$$

Another interesting property is that the probability distribution of the last vertex of a random walk in a expander graph converges rapidly to the uniform distribution. This property is consequence of the following result:

$$\left\|\left(\frac{A_G}{d}\right)^t p - \frac{1}{n}\mathbb{1}\right\| \leq \left(\frac{\max\{|\lambda_2^{\downarrow}(A_G)|, |\lambda_{\min}(A_G)|\}}{d}\right)^t, \quad \text{for each } t \in \mathbb{Z}_{++},$$

and where G is a d-regular graph on n vertices and p is a vector of probabilities.

Spectral Sparsifiers

Let $G = (V, E, w)$ and $H = (V, F, \omega)$ be a weigthed graphs. Let $\varepsilon \geq 0$. The weighted graph H is called a $(1+\varepsilon)$**-spectral sparsifier** of G if

$$x^T L_G x \leq x^T L_H x \leq (1+\varepsilon) x^T L_G x, \quad \text{for each } x \in \mathbb{R}^V,$$

where L_G is the **Laplacian Matrix** of a graph G defined as $L_G := \sum_{ij\in E(G)} w_{ij}(e_i - e_j)(e_i - e_j)^T$.

Cut Sparsifier

Let $G = (V, E, w)$ and $H = (V, F, \omega)$ be weighted graphs. Let $\varepsilon \geq 0$. The weighted graph H is called a $(1+\varepsilon)$**-cut sparsifier** of G if

$$w(\delta_G(S)) \leq \omega(\delta_H(S)) \leq (1+\varepsilon) w(\delta_G(S)), \quad \text{for each } S \subseteq V.$$

Hence, the sum of the weights of the edges of any cut in H is very close to the sum of the weights of the edges of the respective cut in G. By setting $x := \mathbb{1}_S$, one can prove that spectral sparsifiers are a strengthening of cut sparsifiers: if H is a $(1+\varepsilon)$-spectral sparsifier of G, then H is also a $(1+\varepsilon)$-cut sparsifier of G.

Weighted Expanders

We call **Weighted Expanders** graphs that are $(1+\varepsilon)$-spectral sparsifiers of the complete graph, for $\varepsilon > 0$. We consider those sparsifier as **Weighted Expanders** because they satisfy some of the properties that regular expanders satisfy, such as high conductance and the Expander Mixing Lemma. It can be proved that a graph G being a $(1+\varepsilon)$-spectral sparsifier is roughly equivalent to

$$\frac{\lambda_{\max}(L_G)}{\lambda_2^{\uparrow}(L_G)} \leq 1 + \varepsilon,$$

where $\lambda_2^{\uparrow}(L_G)$ denotes the second smallest eigenvalue of L_G, and this ratio is used to measure the quality of the weighted expanders.

Expander Mixing Lemma for Weigthed Expanders

Let $\varepsilon > 0$. Let $G = (V, E, w)$ be a $(1+\varepsilon)$-spectral sparsifier of the weighted complete graph $K_V = (V, \binom{n}{2}, \mathbb{1}d/n)$. Then, for every $S, T \subseteq V$ such that $S \cap T = \emptyset$,

$$\left|w(E(S,T)) - \frac{d}{n}|S||T|\right| \leq d\varepsilon\sqrt{|S||T|}.$$

If we consider a random graph $G(n, p)$ with $p := d/(n-1)$, from this result, one can show that multiplying all the weights of a spectral sparsifier of the complete graph by p implies that the weight of any cut in the sparsifier is close to the expected weight of the respective cut in the random graph.

Theorem of Srivastava and Trevisan

Let $G = (V, E, w)$ be a connected weighted graph with n vertices and $dn/2$ edges such that $\mathrm{girth}(G) \geq 2d^{1/8} + 1$. Suppose that $d \geq 144$. Then

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - O\left(\frac{1}{d^{5/8}}\right) - O\left(\frac{1}{n}\right).$$

The number $1 + 4/\sqrt{d}$ is from approximating the complete graph by a Ramanujan graph on the same vertex set. So

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \leq 1 + \frac{4}{\sqrt{d}} + \frac{8}{d - 2\sqrt{d}}.$$

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