University of São Paulo
Institute of Mathematics and Statistics
Bachelor of Computer Science

Alon Boppana Bound for Non-Regular Expanders

Bruno Hideki Akamine

Final Essay

MAC 499 — Capstone Project

Supervisor: Prof. Marcel K. de Carli Silva

São Paulo
2024

Abstract

Bruno Hideki Akamine. **Alon Boppana Bound for Non-Regular Expanders**. Capstone Project Report (Bachelor). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2024.

The goal of this project is to study generalizations of expander graphs to graphs that are not regular and to prove two Alon-Boppana type bounds for them. The first generalization covered uses the notion of spectral sparsifiers and we consider as expanders the spectral sparsifiers of the complete graph. The second generalization uses the normalized Laplacian matrix of unweighted graphs which do not need to be regular. The proofs of such bounds utilizes some interesting concepts such as non backtracking walks.

Keywords: Spectral Graph Theory. Expander Graphs. Spectral Sparsifier. Laplacian matrix.

Contents

1 Introduction **1**

1.1 Preliminaries on Graph Theory 1

1.2 Algebraic Graph-Theoretical Preliminaries 3

1.3 Motivation 8

2 Weighted Expanders **11**

2.1 Introduction 11

2.2 Preliminaries 14

2.3 Main Result 23

3 Irregular Expanders **37**

3.1 Introduction 37

3.2 Preliminaries 38

3.2.1 Weak Ramanujan Graphs 44

3.2.2 Non-Backtracking Walks 48

3.3 Main Result 50

4 Conclusion **57**

References **59**

Chapter 1

Introduction

Expander graphs are a class of graphs with useful properties, such as forming a single cluster, behaving like random graphs in certain ways, and having the distribution of the last vertex in a random walk converge rapidly to the uniform distribution. Furthermore, expander graphs can be used to construct good error-correcting codes, enabling the retrieval of the original content of a message even if the communication channel is noisy and corrupts the message. Another application is efficiently reducing the error in probabilistic algorithms while using fewer random bits than a naive sampling method.

In the definition of expanders, they are required to be unweighted and regular graphs. Consequently, many studies have explored properties under this traditional definition. One of the most significant results for expander graphs is the result of Alon and Boppana (Nilli, 1991), which bounds the second-largest eigenvalue of the adjacency matrix. This property is key for defining expander graphs and establishing the notion of optimal expanders. A natural question that arises when studying expander graphs is whether it is possible to generalize this notion to non-regular or weighted graphs. In this monograph, two Alon-Boppana type bounds for generalizations of expanders will be studied. The first approach uses spectral sparsifiers to generalize the notion of expanders to weighted graphs, while the second employs the normalized Laplacian matrix to extend the concept to unweighted graphs.

1.1 Preliminaries on Graph Theory

In this section, we will define properties of graphs that will be used later in the text.

An useful operator that will appear in the definition of some of the properties is the **Iverson bracket**, where for an expression P that can be either true or false the Iverson bracket of P is defined as

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

Denote the vector of *all-ones* by $\mathbb{1}$, where the domain is implicit by the context, i.e., if $\mathbb{1} \in \mathbb{R}^V$ then $\mathbb{1}^T e_v = 1$ for each $v \in V$. Also, denote by $\mathbb{1}_U$ the incidence vector of $U \subseteq V$, i.e., one has $\mathbb{1}_U^T e_v = [v \in U]$ for each $v \in V$.

A *weighted graph* is an ordered triple (V, E, w) where (V, E) is a graph and w is a weight function $w\colon E \to \mathbb{R}_{++}$, i.e., the function w assigns weights for each edge of the graph. Note that a graph $G = (V, E)$ can be seen as a weighted graph where all the edges have weight 1, i.e., can be seen as $(V, E, \mathbb{1})$. The complete graph on the vertex set V is denoted by K_V.

Let $G = (V, E, w)$ be a weighted graph. The *neighborhood* of a vertex $v \in V$ is

$$N_G(v) := \{u \in V : u \text{ is adjacent to } v\}.$$

The *combinatorial degree* of a vertex $v \in V$ is $\deg_G(v) := |N(v)|$, where G can be omitted if it is implicit by the context. We define the minimum and maximum combinatorial degree as

$$\delta(G) := \min_{v \in V} \deg_G(v) \qquad \text{and} \qquad \Delta(G) := \max_{v \in V} \deg_G(v).$$

The *girth* of G is the length of a shortest circuit of G, if one exists, and is denoted by $\operatorname{girth}(G)$. The length of a shortest path between two vertices $u, v \in V$ is denoted by $\operatorname{dist}(u, v)$. Let $S \subseteq V$. Denote the *cut* induced by S as

$$\delta_G(S) := \{e \in E : \exists v \in S, \exists u \in V \setminus S, e = uv\},$$

where the index G can be omitted if it is clear in the context. When $|S| = 1$, we abuse notation and use

$$\delta(\{v\}) := \delta(v) \quad \text{for each } v \in V.$$

Denote the *volume* of S as

$$\operatorname{vol}_G(S) := \sum_{v \in S} \deg_G(v).$$

Denote the *weighted degree* of v as $w_v := w^T \mathbb{1}_{\delta(v)}$ for each $v \in V$. We define the minimum and maximum weighted degree of G as

$$\delta_w(G) := \min_{v \in V} w_v \qquad \text{and} \qquad \Delta_w(G) := \max_{v \in V} w_v.$$

Let $\alpha \in \mathbb{R}_{++}$. A *multiple of* G is defined as

$$\alpha G := (V, E, \alpha w).$$

For any set $S \subseteq V$, we denote the *complement* of S as $\overline{S} := V \setminus S$. The *expansion* *ratio* of G is defined as

$$h(G) := \min_{\emptyset \neq S \subset V} \frac{|\delta(S)|}{\min\{|S|, |\overline{S}|\}}.$$

Finally, we can define the expander graphs.

Definition 1. Let $d \in \mathbb{Z}_{++}$ such that $d \geq 2$. Let $(G_n)_{n\in\mathbb{N}}$ be a family of d-regular graphs such that $\lim_{n\to\infty} |V(G_n)| = \infty$. The family $(G_n)_{n\in\mathbb{N}}$ is called a *family of expanders* if there is $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$, one has $h(G_n) \geq \varepsilon$.

By the definition, we can observe that expander graphs are defined as a family of graphs, so we are reffering to a family of graphs whenever we mention expander graphs. Also, from the definition of expander graphs, we see that expander graphs become more sparse as n increases because it has a linear number of edges in relation to the number of vertices of the graph, as every graph in the family is d-regular with d constant.

The *volume* of a set $S \subseteq V$ is

$$\mathrm{vol}_G(S) := \sum_{v\in S} \deg_G(v).$$

The *conductance* of a nonempty set $S \subset V$ is

$$\phi_G(S) := \frac{w^T \mathbb{1}_{\delta(S)}}{\min\{\mathrm{vol}_G(S), \mathrm{vol}_G(\overline{S})\}}.$$

The *conductance* of G is

$$\phi(G) := \min_{\emptyset \neq S \subset V} \phi_G(S).$$

The notion of conductance is useful for indentifying clusters, which are sets $S \subseteq V$ such that the number of edges between vertices in S is considerably larger than the size of $\delta(S)$. Consider a family of d-regular expander graphs, hence $\mathrm{vol}_G(S) = d|S|$ for each $S \subseteq V$. Thus

$$\phi(G) = \min_{\emptyset \neq S \subset V} \frac{w^T \mathbb{1}_{\delta(S)}}{\min\{\mathrm{vol}_G(S), \mathrm{vol}_G(\overline{S})\}} = \frac{|\delta(S)|}{d \min\{|S|, |\overline{S}|\}} = \frac{h(G)}{d}.$$

Therefore, the conductance of expanders is away from zero since their expansion ratio is away from zero. We conclude that expander graphs form a single cluster, i.e., if we take any set of vertices in a expander graph, we observe that the number of edges between vertices in the set is comparable to the number of edges leaving the set.

1.2 Algebraic Graph-Theoretical Preliminaries

In this section, we define some algebraic properties of graphs and prove some results that will be used later.

The *support* of a vector $f \in \mathbb{R}^V$ is

$$\operatorname{supp}(f) := \{i \in V : f_i \neq 0\}.$$

Let $f : V \to \mathbb{R}$ be a function. Let $X \subseteq V$. Denote the *restriction* of f with respect to X as a function $f\restriction_X : V \to \mathbb{R}$ such that $f\restriction_X(x) = f(x)[x \in X]$ for each $x \in V$.

Denote the set of all symmetric $V \times V$ matrices as $\mathbb{S}^V$. A matrix $P \in \mathbb{R}^{V \times V}$ is called a *orthogonal projector* if P is symmetric and idempotent, i.e., if $P = P^T$ and $P = P^2$. Let $U \subseteq V$ be a linear subspace of V, and let $x \in \mathbb{R}^V$. There exist $y \in U$ and $z \in U^\perp$ such that $x = y + z$. A orthogonal projector $P \in \mathbb{S}^V$ projects onto the subspace U if $Px = y$.

Define the function $\lambda^\downarrow : \mathbb{S}^V \to \mathbb{R}^{[n]}$ that sends each $A \in \mathbb{S}^V$ to the vector of eigenvalues of A in non-increasing order. Similarly, define the function $\lambda^\uparrow : \mathbb{S}^V \to \mathbb{R}^{[n]}$ that sends each $A \in \mathbb{S}^V$ to the vector of eigenvalues of A in non-decreasing order.
Theorem 2. Let $A \in \mathbb{S}^V$. Set $n := |V|$. Then there is an orthonormal basis $\{q_1, \dots, q_n\}$ of $\mathbb{R}^V$, all of whose elements are eigenvectors of A, such that

$$A = \sum_{i=1}^{n} \lambda_i^\uparrow(A) q_i q_i^T.$$

From Theorem 2, any symmetric matrix can be decomposed by its eigenvalues and eigenvetors. The next two theorems will show us how to compute the eingenvalues and eigenvectors of a symmetric matrix.
Theorem 3. Let $A \in \mathbb{S}^V$ and set $n := |V|$. Let $k \in [n]$. Let $\{q_i : i \in [k-1]\} \subseteq \mathbb{R}^V$ be an orthonormal set such that $Mq_i = \lambda_i^\downarrow(A) q_i$ for each $i \in [k-1]$. Then

$$\lambda_k^\downarrow = \max\left\{ \frac{x^T A x}{x^T x} : x \in \mathbb{R}^V, x \neq 0, \forall i \in [k-1] \quad x \perp q_i \right\}.$$

Theorem 4. Let $A \in \mathbb{S}^V$ and set $n := |V|$. Let $k \in [n]$. Let $\{q_i : i \in [k-1]\} \subseteq \mathbb{R}^V$ be an orthonormal set such that $Mq_i = \lambda_i^\uparrow(A) q_i$ for each $i \in [k-1]$. Then

$$\lambda_k^\uparrow = \min\left\{ \frac{x^T A x}{x^T x} : x \in \mathbb{R}^V, x \neq 0, \forall i \in [k-1] \quad x \perp q_i \right\}.$$

From Theorem 3 and Theorem 4, all the eigenvalues and eigenvectors of a symmetric matrix can be computed recursively, starting from the largest eigenvalue and its eigenvector (Theorem 3) or starting from the smallest eingevalue and its eigevector (Theorem 4).

The *adjacency matrix* of G is the matrix $A_G \in \mathbb{S}^V$ where, for each $ij \in V \times V$, we have

$$A_G(i, j) = [ij \in E] w_{ij}.$$

The *degree matrix* of G is the matrix $D_G \in \mathbb{S}^V$ where, for each $ij \in V \times V$, we have

$$D_G(i, j) = [i = j] w_i.$$

The *Laplacian matrix* of G is the matrix $L_G \in \mathbb{S}^V$ defined as

$$L_G := \sum_{ij \in E} w_{ij}(e_i - e_j)(e_i - e_j)^T = D_G - A_G.$$

Thus,

$$x^T L_G x = \sum_{ij \in E} w_{ij} x(e_i - e_j)(e_i - e_j)^T x = \sum_{ij \in E} w_{ij}(x_i - x_j)(x_i - x_j) = \sum_{ij \in E} w_{ij}(x_i - x_j)^2,$$

for each $x \in \mathbb{R}^V$. For any $\alpha \in \mathbb{R}_{++}$, we have that $A_{\alpha G} = \alpha A_G$ and $D_{\alpha G} = \alpha D_G$. So $L_{\alpha G} = \alpha L_G$.

Lemma 5. Let $G = (V, E, w)$ be a weighted graph. Let $\alpha \in \mathbb{R}_{++}$. Then

$$\lambda^{\uparrow}(L_{\alpha G}) = \alpha \lambda^{\uparrow}(L_G).$$

Proof. Set $n := |V|$. Let $i \in [n]$. Let x be a $\lambda_i^{\uparrow}(L_G)$-eigenvector of L_G. Then

$$L_{\alpha G} x = \alpha L_G x = \alpha \lambda_i^{\uparrow}(L_G) x.$$ □

Let $A, B \in \mathbb{S}^V$. We say that A is *positive semidefinite* if $x^T A x \geq 0$ for each $x \in \mathbb{R}^V$. We write $A \succcurlyeq B$ when $A - B$ is positive semidefinite. Thus, $A \succcurlyeq 0$ if and only if A is positive semidefinite.

Theorem 6. Let $A, B \in \mathbb{S}^V$ such that $A \succcurlyeq B$. Then $\lambda^{\uparrow}(A) \geq \lambda^{\uparrow}(B)$.

Lemma 7. Let $G = (V, E, w)$ be a weighted graph. Then L_G is positive semidefinite.

Proof. Let $x \in \mathbb{R}^V$. Then

$$x^T L_G x = \sum_{ij \in E} w_{ij}(x_i - x_j)^2.$$

Note that for each $ij \in E$ we have that $w_{ij} \geq 0$ and $(x_i - x_j)^2 \geq 0$, hence $x^T L x \geq 0$. □

The *Normalized Laplacian matrix* is the matrix $\mathcal{L}_G \in \mathbb{S}^V$ defined as

$$\mathcal{L}_G := D_G^{-1/2} L_G D_G^{-1/2}.$$

Lemma 8. Let $G = (V, E, w)$ be a weighted graph. Then $\mathcal{L}_G$ is positive semidefinite.

Proof. Let $x \in \mathbb{R}^V$. Then

$$x^T \mathcal{L}_G x = x^T D_G^{-1/2} L_G D_G^{-1/2} x.$$

Note that $D^{-1/2} x \in \mathbb{R}^V$. Since $D_G \in \mathbb{S}^V$ and using Lemma 7,

$$x^T \mathcal{L}_G x = x^T D_G^{-1/2} L_G D_G^{-1/2} x = (D_G^{-1/2} x)^T L_G D_G^{-1/2} x \geq 0.$$ □

From Theorem 2, both the Laplacian matrix and the normalized Laplacian matrix of a graph can be decomposed into their eigenvalues and eigenvectors. The next lemma will show the relationship between a graph being positive semidefinite and its eigenvalues.

Lemma 9. Let $A \in \mathbb{S}^V$. Then A is positive semidefinite if and only if $\lambda^{\uparrow}(A) \geq 0$.

Proof. If A is positive semidefinite, it holds that $x^T Ax \geq 0$ for each $x \in \mathbb{R}^V$. Let $x \in \mathbb{R}^V$ and let $\alpha \in \mathbb{R}$ such that x is an α-eigenvetor of A. Suppose $\alpha < 0$, hence

$$x^T Ax = x^T(Ax) = x^T \alpha x = \alpha x^T x = \alpha \|x\|^2.$$

Since $\|x\|^2 > 0$, we have that $x^T Ax < 0$, a contradiction.

Suppose $\lambda^{\uparrow}(A) \geq 0$. Let $x \in \mathbb{R}^V$ and set $n := |V|$. By Theorem 2, there is an orthonormal basis $\{q_1, \dots, q_n\}$ of $\mathbb{R}^V$, all of whose elements are eigenvectors of A, such that $A = \sum_{i=1}^{n} \lambda_i^{\uparrow}(A) q_i q_i^T$. Hence

$$x^T Ax = \sum_{i=1}^{n} \lambda_i^{\uparrow}(A) x^T q_i q_i^T x = \sum_{i=1}^{n} \lambda_i^{\uparrow}(A)(x^T q_i)^2.$$

Since the square of any real number is nonnegative and $\lambda_i^{\uparrow}(A) \geq 0$ for every $i \in [n]$, we conclude that $x^T Ax$ is a sum of nonnegative terms and so $x^T Ax \geq 0$. □

Lemma 9, together with Lemma 7 and Lemma 8, tells us that all the eigenvalues of the laplacian matrix and the normalized laplacian matrix of a graph are nonnegative.

Theorem 10. Let $G = (V, E, w)$ be a weighted graph. Then

$$\mathrm{Null}(L_G) = \mathrm{span}\{\mathbb{1}_C : C \subseteq V \text{ is a component of } G\}.$$

Corollary 11. Let $G = (V, E, w)$ be a weighted graph. Then $\lambda_1^{\uparrow}(L_G) = 0$ and $\mathbb{1}$ is a 0 - eigenvector of L_G.

Proof. Immediate from Theorem 10. □

Corollary 12. Let (G, E, w) be a weighted graph. Then G is connected if and only if $\lambda_2^{\uparrow}(L_G) > 0$.

Proof. Immediate from Theorem 10. □

Corollary 12 will be important in chapter 2, since the graphs we will consider in this chapter must be connected. Otherwise, we would have a division by 0 in the denominator.

Lemma 13. Let $G = (V, E, w)$ be a weighted graph. Then $\lambda_1^{\uparrow}(\mathcal{L}_G) = 0$ and $D_G^{1/2}\mathbb{1}$ is a 0-eigenvetor of $\mathcal{L}_G$.

Proof. By the definition of $\mathcal{L}_G$,

$$\mathcal{L}_G D^{1/2}\mathbb{1} = D^{-1/2} L_G D^{-1/2} D^{1/2}\mathbb{1} = D^{-1/2} L_G \mathbb{1}.$$

Using Corollary 11,

$$\mathcal{L}_G D^{1/2}\mathbb{1} = D^{-1/2} L_G \mathbb{1} = D^{-1/2} 0 = 0.$$

By Lemma 8, it follows that $\lambda_1^{\uparrow}(\mathcal{L}_G) = 0$. □

Corollary 14. Let $G = (V, E, w)$ be a weighted graph. Then, G is connected if and only if $\lambda_2^{\uparrow}(\mathcal{L}_G) > 0$.

Proof. Immediate from Corollary 12. $\square$

Corollary 14 will be important in chapter 3, because if G is not connected, then the bound being proved in this chapter is trivial.

Let $X \in \mathbb{R}^{V \times V}$ be a square matrix. The *trace* of X is

$$\mathrm{Tr}(X) = \sum_{i \in V} X_{ii}.$$

Lemma 15. Let $X \in \mathbb{S}^V$. Then

$$\mathrm{Tr}(X) = \mathbb{1}^T \lambda^{\uparrow}(X).$$

The next two lemma are basic facts of linear algebra but will be useful in some proofs throughout the text.

Lemma 16. Let U be a linear subspace of V. Let B_U be a orthonormal basis of U. Then

$$\sum_{b \in B_U} bb^T = \mathrm{Proj}_U.$$

Proof. First, we prove that the sum is orthogonal,

$$\Big(\sum_{b \in B_U} bb^T\Big)^T = \sum_{b \in B_U} \big(bb^T\big)^T = \sum_{b \in B_U} bb^T.$$

By assumption, we have $b_1^T b_2 = [b_1 = b_2]$ for each $b_1, b_2 \in B_U$. So,

$$\Big(\sum_{b \in B_U} bb^T\Big)^T \Big(\sum_{b \in B_U} bb^T\Big) = \sum_{b_1 \in B_U} \sum_{b_2 \in B_U} b_1 b_1^T b_2 b_2^T = \sum_{b_1 \in B_U} \sum_{b_2 \in B_U} b_1 [b_1 = b_2] b_2^T = \sum_{b \in B_U} bb^T.$$

Let $x \in V$. There exist $y \in U$ and $z \in U^{\perp}$ such that $x = y + z$. We can extend B_U to be a orthonormal basis of V. Take the extended orthogonal basis B_V, note that $B_{U^\perp} := B_V \setminus B_U$ is a orthonormal basis of $U^{\perp}$. Hence, there are coeficients $(\alpha_b)_{b \in B_U}$ such that $y = \sum_{b \in B_U} \alpha_b bb^T$, and there are coeficients $(\beta_b)_{b \in B_{U^\perp}}$ such that $z = \sum_{b \in B_{U^\perp}} \beta_b bb^T$. Since $b_1^T b_2 = 0$ for each $b_1 \in B_U$ and for each $b_2 \in B_{U^\perp}$,

$$\begin{aligned}
\Big(\sum_{b \in B_U} bb^T\Big) x &= \Big(\sum_{b \in B_U} bb^T\Big)(y + z) = \Big(\sum_{b \in B_U} bb^T\Big) y + \Big(\sum_{b \in B_U} bb^T\Big) z \\
&= \sum_{b_1 \in B_U} \sum_{b_2 \in B_U} \alpha_{b_2} b_1 b_1^T b_2 b_2^T + \sum_{b_1 \in B_U} \sum_{b_2 \in B_{U^\perp}} \beta_{b_2} b_1 b_1^T b_2 b_2^T \\
&= \sum_{b_1 \in B_U} \sum_{b_2 \in B_U} \alpha_{b_2} b_1 [b_1 = b_2] b_2^T + 0 \\
&= \sum_{b \in B} \alpha_b bb^T = y.
\end{aligned}$$

$\square$

Lemma 17. Let $f \in \mathbb{R}^V$. Then

$$\|f\|^2 = \|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\|^2 + \|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\|^2.$$

Proof. Note that $\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f = \left(I - \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}}\right) f$. Since a projector is symmetric and idempotent:

$$\begin{aligned}\left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\right)^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} f &= f^T \mathrm{Proj}^T_{\mathrm{span}\{\mathbb{1}\}}\left(I - \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}}\right) f \\ &= f^T\left(\mathrm{Proj}^T_{\mathrm{span}\{\mathbb{1}\}} - \mathrm{Proj}^2_{\mathrm{span}\{\mathbb{1}\}}\right) f \\ &= f^T(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} - \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}}) f = 0\end{aligned}$$

So,

$$\begin{aligned}\|f\|^2 &= \|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f + \mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\|^2 \\ &= \left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f + \mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right)^T\left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f + \mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right) \\ &= \left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\right)^T\left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\right) + \left(\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right)^T\left(\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right) + 2\left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\right)^T\left(\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right) \\ &= \|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\|^2 + \|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\|^2.\end{aligned}$$

□

1.3 Motivation

In section 1.1, we defined expander graphs and showed one of their properties: that the vertices of expander graphs lie in a single cluster. In this section, we explore more properties and explain in more details some of the things discussed at the beggining of the chapter. A result that was very important due to Cheeger and Buser, which bounds the expansion ratio of a graph using the spectral gap of the adjacency matrix of the graph, is the following.

Theorem 18. (see [Hoory *et al.*, 2006, Theorem2.4]) Let G be a d-regular graph. Then

$$\frac{d - \lambda_2^{\downarrow}(A_G)}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2^{\downarrow}(A_G))}.$$

Recall that d is the largest eingevalue of a d-regular graph. Hence, the value $d - \lambda_2^{\downarrow}(A_G)$ for any d-regular graph is called the *espectral gap* of G. From Theorem 18, one can show that the definition of expanders can use the spectral gap instead of the expansion ratio.

Corollary 19. Let $d \in \mathbb{Z}_{++}$ such that $d \geq 2$. Let $(G_n)_{n \in \mathbb{N}}$ be a family of d-regular graphs such that $\lim\limits_{n\to\infty} = \infty$. The family $(G_n)_n \in \mathbb{N}$ is called a *family of expanders* if and only if there is $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$, one has $d - \lambda_2^{\downarrow}(A_{G_n}) \geq \varepsilon$.

Alon and Boppana bounded below the second largest eigenvalue of the adjacency matrix of d-regular graphs, hence they bounded the spectral gap of d-regular graphs.

Theorem 20. ([Nilli, 1991]) Let $G = (V, E)$ be a d-regular graph with $d \geq 2$. Then

$$\lambda_2^{\downarrow}(A_G) \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{\frac{\operatorname{diam}(G)}{2} - 1}.$$

From Theorem 20 and the updated definition in Corollary 19, one can think of optimal expanders. The optimal expanders, which are called *Ramanujan graphs*, use the bound proved by Alon and Boppana.

Definition 21. Let $G = (V, E)$ be a d-regular graph with n vertices and with $d \geq 2$. The graph G is called *Ramanujan graph* if, for each $i \in [n]$, one has

$$|\lambda_i^{\downarrow}(A_G)| \leq 2\sqrt{d-1} \quad \text{or} \quad |\lambda_i^{\downarrow}(A_G)| = d.$$

There are some deterministic constructions of familys of Ramanujan graphs. One recent example is a construction that uses stable polynomials to create bipartite Ramanujan graphs [Marcus *et al.*, 2015].

An important property of expander graphs is the Expander Mixing Lemma.

Lemma 22. (Expander Mixing Lemma) Let $G = (V, E)$ be a d-regular graph with $n \geq 2$ vertices and $d \geq 1$. Then, for all $S, T \subseteq V$,

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \max\{|\lambda_2^{\downarrow}(A_G)|, |\lambda_{min}(A_G)|\}\sqrt{|S||T|},$$

where $E(S, T)$ is the set of edges with one end in S and the other in T.

From the Expander Mixing Lemma (Lemma 22), one has that the number of edges between any two set of vertices S and T in an expander graph is approximately the number of expected edges between any two sets of size $|S|$ and $|T|$ in a random graph with the same number of vertices and with the same number of expected edges of the expander graph. Hence, expander graphs behave as random graphs with the same number of vertices and the same expected number of edges.

Another interesting property of expanders is about random walks in the expander graphs.

Lemma 23. Let $G = (V, E)$ be a d-regular graph. Let p be a vector of probabilities. Then

$$\left\| \left(\frac{A_G}{d}\right)^t p - \frac{1}{n}\mathbb{1} \right\| \leq \left(\frac{\max\{|\lambda_2^{\downarrow}(A_G)|, |\lambda_{min}(A_G)|\}}{d} \right)^t.$$

From Lemma 23, one has that random walks in expander graphs converge rapidly to the uniform distribution. Hoory, Linial, and Wigderson [Hoory *et al.*, 2006] discussed many other applications and properties of expanders, such as efficiently reducing the error in probabilistic algorithms.

Throughout this chapter, the main focus was to introduce some notations and results, and familiarize the reader to the traditional notion of expander graphs, which are d-regular graphs. In the next chapters, we introduce two different generalizations of expanders graphs and prove Alon Boppana bounds for these generalizations.

Chapter 2

Weighted Expanders

2.1 Introduction

One of the approaches to generalize the notion of expander graphs beyond regular graphs uses spectral sparsifiers of complete graphs.

Definition 24. Let $G = (V, E, w)$ and $H = (V, F, \omega)$ be weighted graphs. Let $\varepsilon \geq 0$. The weighted graph H is called a $(1+\varepsilon)$-*spectral sparsifier* of G if

$$L_G \preccurlyeq L_H \preccurlyeq (1+\varepsilon) L_G.$$

Note that, when $\varepsilon = 0$, this can only hold if the graph G is the graph H. We also say that H is a $(1 + \varepsilon)$-*approximation* of G.

Batson, Spielman, and Srivastava [Batson *et al.*, 2012] proved that, for every graph G with average degree at most $2d$, there exists a weighted subgraph of G that is a $(1+\varepsilon)$-spectral sparsifier of G, where

$$1 + \varepsilon := \frac{d + 2\sqrt{d-1}}{d - 2\sqrt{d-1}} = 1 + \frac{4\sqrt{d-1}}{d - 2\sqrt{d-1}},$$

by showing a deterministic algorithm for constructing such weighted graphs. In the context of approximating complete graphs, we can observe some properties of expander graphs in their sparsifiers, such as the following version of the Expander Mixing Lemma.

Lemma 25 ([Batson *et al.*, 2012, Lemma 4.1])**.** Let $\varepsilon > 0$. Let $G = (V, E, w)$ be a weighted connected graph. Suppose that G is a $(1+\varepsilon)$-spectral sparsifier of K_V. Then

$$\left| \mathbb{1}_S^T A_G \mathbb{1}_T - \left(1 + \frac{\varepsilon}{2}\right)|S||T| \right| \leq n\left(\frac{\varepsilon}{2}\right)\sqrt{|S||T|} \quad \text{for each } S, T \subseteq V \text{ s.t. } S \cap T = \emptyset.$$

Thus, it is reasonable to think of sparsifiers of complete graphs as expanders that are weighted and not necessarily regular. Hence, we consider this class of expanders as *weighted expanders*. Another way to analyze a weighted expander G is through the ratio $\lambda_n^{\uparrow}(L_G)/\lambda_2^{\uparrow}(L_G)$, referred to as the finite condition number of the Laplacian, which is a fundamental object of study in Numerical Linear Algebra, and has a strong connection to sparsification as shown below.

Lemma 26. Let $\varepsilon \in \mathbb{R}_{++}$. Let $G = (V, E, w)$ be a weighted graph with n vertices. If G is a $(1+\varepsilon)$-spectral sparsifier of K_V, then

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \leq 1 + \varepsilon. \tag{2.1}$$

Proof. Since G is a $(1+\varepsilon)$-sparsifier of K_V,

$$L_{K_V} \preccurlyeq L_G \qquad \text{and} \tag{2.2a}$$
$$L_G \preccurlyeq (1+\varepsilon) L_{K_V}. \tag{2.2b}$$

By Theorem 6, (2.2a) and (2.2b),

$$0 < \lambda_2^{\uparrow}(L_{K_V}) \leq \lambda_2^{\uparrow}(L_G) \quad \text{and} \tag{2.3a}$$
$$\lambda_n^{\uparrow}(L_G) \leq \lambda_n^{\uparrow}((1+\varepsilon) L_{K_V}). \tag{2.3b}$$

Note that $\lambda_2^{\uparrow}(L_{K_V}) = \cdots = \lambda_n^{\uparrow}(L_{K_V}) = n$. So, one has that $\lambda_n^{\uparrow}((1+\varepsilon)L_{K_V}) = (1+\varepsilon)n$. Hence,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \leq \frac{\lambda_n^{\uparrow}((1+\varepsilon)L_{K_V})}{\lambda_2^{\uparrow}(L_{K_V})} = \frac{(1+\varepsilon)n}{n} = 1+\varepsilon,$$

where (2.3a) and (2.3b) are used to reach the first inequality. □

Lemma 27. Let $\varepsilon \in \mathbb{R}_{++}$. Let $G = (V, E, w)$ be a weighted graph. If

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \leq 1 + \varepsilon, \tag{2.4}$$

then αG is a $(1+\varepsilon)$–spectral sparsifier of K_V, where $\alpha := n/\lambda_2^{\uparrow}(L_G)$.

Proof. Set $n := |V|$. Note that

$$L_{K_n} = nI - J = n\Big(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\Big) = n\Big(I - \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}}\Big) = n\mathrm{Proj}_{\{\mathbb{1}\}^{\perp}}. \tag{2.5}$$

Using Lemma 5, we have that $\lambda^{\uparrow}(L_{\alpha G}) = \frac{n}{\lambda_2^{\uparrow}(L_G)} \lambda^{\uparrow}(L_G)$. Since $\lambda_1^{\uparrow}(L_G) = \lambda_1^{\uparrow}(L_{\alpha G}) = 0$, we can decompose the Laplacian as follows:

$$L_{\alpha G} = \sum_{i=1}^{n} \lambda_i^{\uparrow}(L_{\alpha G}) u_i u_i^T = \sum_{i=2}^{n} \lambda_i^{\uparrow}(L_{\alpha G}) u_i u_i^T,$$

where $\{u_1, \ldots, u_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Since $\lambda_2^{\uparrow}(L_{\alpha G}) \leq \lambda_i^{\uparrow}(L_{\alpha G})$, for each $i \in [n] \setminus \{1\}$,

$$x^T L_{\alpha G} x = x^T\Big(\sum_{i=2}^{n} \lambda_i^{\uparrow}(L_{\alpha G}) u_i u_i^T\Big)x \geq x^T\Big(\sum_{i=2}^{n} \lambda_2^{\uparrow}(L_{\alpha G}) u_i u_i^T\Big)x = \lambda_2^{\uparrow}(L_{\alpha G}) x^T\Big(\sum_{i=2}^{n} u_i u_i^T\Big)x.$$

Using Lemma 16,

$$
\begin{aligned}
x^T L_{\alpha G} x &\geq \lambda_2^{\uparrow}(L_{\alpha G}) x^T \Big(\sum_{i=2}^{n} u_i u_i^T \Big) x = \lambda_2^{\uparrow}(L_{\alpha G}) x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x \\
&= \frac{n}{\lambda_2^{\uparrow}(L_G)} \lambda_2^{\uparrow}(L_G) x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x = n x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x.
\end{aligned} \tag{2.6}
$$

Additionaly, since $\lambda_n^{\uparrow}(L_G) \geq \lambda_i^{\uparrow}(L_G)$ for each $i \in [n]$,

$$
x^T L_{\alpha G} x = x^T \Big(\sum_{i=2}^{n} \lambda_i^{\uparrow}(L_{\alpha G}) u_i u_i^T \Big) x \leq x^T \Big(\sum_{i=2}^{n} \lambda_n^{\uparrow}(L_{\alpha G}) u_i u_i^T \Big) x = \lambda_n^{\uparrow}(L_{\alpha G}) x^T \Big(\sum_{i=2}^{n} u_i u_i^T \Big).
$$

Using Lemma 16,

$$
\begin{aligned}
x^T L_{\alpha G} x &\leq \lambda_n^{\uparrow}(L_{\alpha G}) x^T \Big(\sum_{i=2}^{n} u_i u_i^T \Big) x = \lambda_n^{\uparrow}(L_{\alpha G}) x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x \\
&= \frac{n}{\lambda_2^{\uparrow}(L_G)} \lambda_n^{\uparrow}(L_G) x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x = n \frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x.
\end{aligned} \tag{2.7}
$$

So using (2.6) and (2.5),

$$
x^T (L_{\alpha G} - L_{K_n}) x \geq n x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x - n x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x = 0.
$$

Hence, $L_{K_n} \preccurlyeq L_{\alpha G}$. Using (2.7), (2.4) and (2.5),

$$
\begin{aligned}
x^T ((1+\varepsilon) L_{K_n} - L_{\alpha G}) x &\geq (1+\varepsilon) n x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x - n \frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x \\
&\geq n \frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x - n \frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} x^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} x \\
&= 0.
\end{aligned} \tag{2.8}
$$

So, we have that $L_{\alpha G} \preccurlyeq (1+\varepsilon) L_{K_n}$. □

One of the most interesting family of expander graphs, using the traditional definiton of expanders, are the Ramanujan Graphs. If G is a Ramanujan Graph (Definition 21) then using the bound proved by Alon and Boppana [Nilli, 1991] give us

$$
\begin{aligned}
\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} &\leq \frac{d + 2\sqrt{d-1}}{d - 2\sqrt{d-1}} = \frac{d - 2\sqrt{d-1} + 4\sqrt{d-1}}{d - 2\sqrt{d-1}} \\
&= 1 + \frac{4\sqrt{d-1}}{d - 2\sqrt{d-1}} \leq 1 + \frac{4\sqrt{d}}{d - 2\sqrt{d}} \\
&= 1 + \frac{4}{\sqrt{d}} + \frac{8}{d - 2\sqrt{d}}
\end{aligned}
$$

So a question that has been considered recently is whether the bound can be generalized to weighted expanders and if this bound remains optimal for a broader category of expanders.

2.2 Preliminaries

Lemma 28. Let $G = (V, E, w)$ be a weighted graph with n vertices. Let $\kappa \geq 1$. If G is a κ-spectral sparsifier of K_V. Then $\delta_w(G) \geq n-1$ and $\Delta_w(G) \leq \kappa(n-1)$.

Proof. Let $v \in V$. Note that

$$e_v^T L_{K_V} e_v = e_v^T (nI - J) e_v = n e_v^T I e_v - e_v^T J e_v = n - 1,$$

and

$$\kappa e_v^T L_{K_V} e_v = \kappa e_v^T (nI - J) e_v = \kappa(n e_v^T I e_v - e_v^T J e_v) = \kappa(n-1).$$

Since G is a κ-spectral sparsifier of K_V, we have that $L_{K_V} \preccurlyeq L_G \preccurlyeq \kappa L_{K_V}$. Hence,

$$n - 1 = e_v^T L_{K_V} e_v \leq e_v^T L_G e_v,$$

and

$$e_v^T L_G e_v \leq \kappa e_v^T L_{K_n} e_v = \kappa(n-1).$$

But we have that

$$e_v^T L_G e_v = e_v^T D e_v - e_v^T A_G e_v = w_v - 0 = w_v.$$

So, we conlude that $n - 1 \leq e_v^T L_G e_v = w_u$ and $w_u = e_v^T L_G e_v \leq \kappa(n-1)$. □

Lemma 29. Let $G = (V, E, w)$ be a weighted graph. Let $x \in \mathbb{R}^V$. Then

$$\mathrm{Proj}_{\{\mathbb{1}\}^\perp}^T L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp} = L_G.$$

Proof. It is sufficient to prove that $L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp} = L_G$, because L_G is orthogonal. So,

$$\mathrm{Proj}_{\{\mathbb{1}\}^\perp}^T L_G = \left(L_G^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp}\right)^T = \left(L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp}\right)^T.$$

Since $\mathrm{Proj}_{\{\mathbb{1}\}^\perp} = (I - \mathbb{1}\mathbb{1}^T / \mathbb{1}^T \mathbb{1}) = (I - \mathbb{1}\mathbb{1}^T / n)$ and by Corollary 11,

$$L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp} = L_G\left(I - \frac{\mathbb{1}\mathbb{1}^T}{n}\right) = L_G - \frac{L_G \mathbb{1}\mathbb{1}^T}{n} = L_G - 0 = L_G.$$ □

Theorem 30 (see [Batson *et al.*, 2012, Proposition 4.2])**.** Let $G = (V, E, w)$ be a connected weighted graph with n vertices and let d be the combinatorial degree of some vertex. Let $\kappa \geq 1$. If G is a κ-spectral sparsifier of K_V. Then

$$\kappa \geq 1 + \frac{2}{\sqrt{d}} - \frac{8\sqrt{d}}{n}.$$

Proof. There exist $v \in V$ of combinatorial degree d. Define the function $f : V \to \mathbb{R}$ as

$$f(u) = \begin{cases} 1 & \text{if } u = v, \\ 1/\sqrt{d} & \text{if } u \in N(v), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for each } u \in V.$$

Also, define $g : V \to \mathbb{R}$ as

$$g(u) = \begin{cases} 1 & \text{if } u = v, \\ -1/\sqrt{d} & \text{if } u \in N(v), \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for each } u \in V.$$

Consider the induced subrgraph $G' := G\big[N(v) \cup \{v\}\big]$. Set

$$\overline{w}_u := \sum_{x \in N(u) \setminus (N(v) \cup \{v\})} w_{ux} \quad \text{for each } u \in N(v).$$

Note that the functions f and g are constant over $N(v)$. Hence,

$$\begin{aligned}
f^T L_G f &= \sum_{u \in N(v)} \left(w_{vu}(f(v) - f(u))^2 + \sum_{x \in N(u) \setminus (N(v) \cup \{v\})} w_{ux}(f(u) - f(x))^2 \right) \\
&= \sum_{u \in N(v)} \left(w_{vu}(1 - 1/\sqrt{d})^2 + \sum_{x \in N(u) \setminus (N(v) \cup \{v\})} w_{ux}(1/\sqrt{d} - 0)^2 \right) \\
&= \sum_{u \in N(v)} \left(w_{uv} - 2w_{uv}/\sqrt{d} + w_{uv}/d + \overline{w}_u/d \right) \\
&= \sum_{u \in N(v)} \left(w_{uv} + (w_{uv} + \overline{w}_u)/d - 2w_{uv}/\sqrt{d} \right) \\
&= \sum_{u \in N(v)} w_{uv} + \sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d - \frac{1}{\sqrt{d}} 2 \sum_{u \in N(v)} w_{uv} \\
&= 1 - \frac{1}{\sqrt{d}} \frac{2 \sum_{u \in N(v)} w_{uv}}{\sum_{u \in N(v)} w_{uv} + \sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d},
\end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
g^T L_G g &= \sum_{u \in N(v)} \left(w_{uv}(f(u) - f(v))^2 + \sum_{x \in N(u) \setminus (N(v) \cup \{v\})} w_{ux}(f(u) - f(x))^2 \right) \\
&= \sum_{u \in N(v)} \left(w_{uv}(1 + 1/\sqrt{d})^2 + \sum_{x \in N(u) \setminus (N(v) \cup \{v\})} w_{ux}(-1/\sqrt{d} - 0)^2 \right) \\
&= \sum_{u \in N(v)} \left(w_{uv} + 2w_{uv}/\sqrt{d} + w_{uv}/d + \overline{w}_u/d \right) \\
&= \sum_{u \in N(v)} \left(w_{uv} + (w_{uv} + \overline{w}_u)/d + 2w_{uv}/\sqrt{d} \right) \\
&= \sum_{u \in N(v)} w_{uv} + \sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d + \frac{1}{\sqrt{d}} 2 \sum_{u \in N(v)} w_{uv} \\
&= 1 + \frac{1}{\sqrt{d}} \frac{2 \sum_{u \in N(v)} w_{uv}}{\sum_{u \in N(v)} w_{uv} + \sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d}.
\end{aligned}
\tag{2.10}
$$

Note that

$$
\frac{2 \sum_{u \in N(v)} w_{uv}}{\sum_{u \in N(v)} w_{uv} + \sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d} = \frac{2}{1 + \frac{\sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d}{\sum_{u \in N(v)} w_{uv}}}
$$

By Lemma 28, $\delta_w(G) \geq n - 1$ and $\Delta_w(G) \leq \kappa (n - 1)$. So, we have that $\sum_{u \in N(v)} w_{uv} = w_v \geq n - 1$. Also,

$$
\frac{1}{d} \sum_{u \in N(v)} (w_{uv} + \overline{w}_u) \leq \frac{1}{d} \sum_{u \in N(v)} w_u \leq \frac{1}{d} \sum_{u \in N(v)} \kappa(n-1) = \frac{1}{d} d\kappa(n-1) = \kappa(n-1).
$$

Hence,

$$
\frac{\sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d}{\sum_{u \in N(v)} w_{uv}} \leq \frac{\kappa(n-1)}{n-1} = \kappa.
$$

And so,

$$
\frac{2}{1 + \frac{\sum_{u \in N(v)} (w_{uv} + \overline{w}_u)/d}{\sum_{u \in N(v)} w_{uv}}} \geq \frac{2}{1 + \kappa}.
$$

From (2.9) and (2.10)

$$
, f^T L_G f \leq 1 - \frac{1}{\sqrt{d}} \frac{2}{1 + \kappa}, \tag{2.11}
$$

and

$$
g^T L_G g \geq 1 + \frac{1}{\sqrt{d}} \frac{2}{1 + \kappa}. \tag{2.12}
$$

Since $\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} = \mathbb{1}\mathbb{1}^T/\mathbb{1}^T\mathbb{1}$ and by Lemma 17, we have that

$$\begin{aligned}\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right\|^2 &= \|f\|^2 - \left\|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\right\|^2 = 2 - \left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f\right)^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f \\ &= 2 - f^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}}^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f = 2 - f^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f \\ &= 2 - f^T \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}} f = 2 - \frac{(f^T\mathbb{1})^2}{n} = 2 - \frac{(1+\sqrt{d})^2}{n},\end{aligned}$$

and

$$\begin{aligned}\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} g\right\|^2 &= \|g\|^2 - \left\|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} g\right\|^2 = 2 - \left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} g\right)^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} g \\ &= 2 - g^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}}^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} g = 2 - g^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} g \\ &= 2 - g^T \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}} g = 2 - \frac{(g^T\mathbb{1})^2}{n} = 2 - \frac{(1-\sqrt{d})^2}{n}.\end{aligned}$$

So,

$$\begin{aligned}\frac{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right\|^2}{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} g\right\|^2} &= \frac{2 - \frac{(1+\sqrt{d})^2}{n}}{2 - \frac{(1-\sqrt{d})^2}{n}} = \frac{2n - (1+\sqrt{d})^2}{2n - (1-\sqrt{d})^2} = \frac{2n - 1 - 2\sqrt{d} - d}{2n - 1 + 2\sqrt{d} - d} \\ &= 1 - \frac{4\sqrt{d}}{2n - 1 + 2\sqrt{d} - d}.\end{aligned}$$

Since d is the degree of v, we have that $d \le n-1$, which implies that $d+1 \le n$. So,

$$2n - 1 + 2\sqrt{d} - d = 2n + 2\sqrt{d} - (d+1) \ge 2n + 2\sqrt{d} - n = n + 2\sqrt{d} \ge n.$$

Hence,

$$\frac{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right\|^2}{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} g\right\|^2} = 1 - \frac{4\sqrt{d}}{2n - 1 + 2\sqrt{d} - d} \ge 1 - \frac{4\sqrt{d}}{n}. \tag{2.13}$$

Combining (2.11), (2.12) and (2.13) and using Lemma 26 and Lemma 29,

$$\begin{aligned}\kappa \ge \frac{\lambda_n^\uparrow(L_G)}{\lambda_2^\uparrow(L_G)} &\ge \frac{\left(\mathrm{Proj}_{\{\mathbb{1}\}^\perp} g\right)^T L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp} g}{\left(\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right)^T L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp} f} \frac{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right\|^2}{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} g\right\|^2} \\ &= \frac{g^T L_G g}{f^T L_G f} \frac{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f\right\|^2}{\left\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} g\right\|^2} \ge \frac{1 + \frac{1}{\sqrt{d}}\frac{2}{1+\kappa}}{1 - \frac{1}{\sqrt{d}}\frac{2}{1+\kappa}} \left(1 - \frac{4\sqrt{d}}{n}\right).\end{aligned}$$

Rearranging the terms,

$$\begin{aligned}\kappa\left(1 - \frac{1}{\sqrt{d}}\frac{2}{1+\kappa}\right) &\ge \left(1 + \frac{1}{\sqrt{d}}\frac{2}{1+\kappa}\right)\left(1 - \frac{4\sqrt{d}}{n}\right) \\ &= 1 + \frac{1}{\sqrt{d}}\frac{2}{1+\kappa} - \frac{4\sqrt{d}}{n} - \frac{4\sqrt{d}}{n}\frac{1}{\sqrt{d}}\frac{2}{1+\kappa}.\end{aligned}$$

Since $\kappa \geq 1$ and $d \geq 1$, we have that $0 < \frac{1}{\sqrt{d}}\frac{2}{1+\kappa} \leq 1$. So,

$$\frac{4\sqrt{d}}{n}\frac{1}{\sqrt{d}}\frac{2}{1+\kappa} \leq \frac{4\sqrt{d}}{n},$$

whence

$$\begin{aligned}\kappa\left(1 - \frac{1}{\sqrt{d}}\frac{2}{1+\kappa}\right) &\geq 1 + \frac{1}{\sqrt{d}}\frac{2}{1+\kappa} - \frac{4\sqrt{d}}{n} - \frac{4\sqrt{d}}{n}\frac{1}{\sqrt{d}}\frac{2}{1+\kappa}\\ &\geq 1 + \frac{1}{\sqrt{d}}\frac{2}{1+\kappa} - \frac{4\sqrt{d}}{n} - \frac{4\sqrt{d}}{n}\\ &= 1 + \frac{1}{\sqrt{d}}\frac{2}{1+\kappa} - \frac{8\sqrt{d}}{n}.\end{aligned}$$

Set $\gamma := 1 + \kappa$. Hence,

$$(\gamma - 1)\left(1 - \frac{1}{\sqrt{d}}\frac{2}{\gamma}\right) \geq 1 + \frac{1}{\sqrt{d}}\frac{2}{\gamma} - \frac{8\sqrt{d}}{n}.$$

Multiplying all the terms by γ,

$$(\gamma - 1)\left(\gamma - \frac{2}{\sqrt{d}}\right) \geq \gamma + \frac{2}{\sqrt{d}} - \gamma\frac{8\sqrt{d}}{n}.$$

Rearranging the inequality,

$$\begin{aligned}&(\gamma - 1)\left(\gamma - \frac{2}{\sqrt{d}}\right) \geq \gamma + \frac{2}{\sqrt{d}} - \gamma\frac{8\sqrt{d}}{n}\\ \Rightarrow &\gamma^2 - \gamma\frac{2}{\sqrt{d}} - \gamma + \frac{2}{\sqrt{d}} \geq \gamma + \frac{2}{\sqrt{d}} - \gamma\frac{8\sqrt{d}}{n}\\ \Rightarrow &\gamma^2 - \gamma\frac{2}{\sqrt{d}} - 2\gamma + \gamma\frac{8\sqrt{d}}{n} \geq 0\\ \Rightarrow &\gamma\left(\gamma - \frac{2}{\sqrt{d}} - 2 + \frac{8\sqrt{d}}{n}\right) \geq 0.\end{aligned}$$

Since $\gamma = 1 + \kappa > 0$,

$$\gamma - \frac{2}{\sqrt{d}} - 2 + \frac{8\sqrt{d}}{n} \geq 0 \Rightarrow \gamma \geq 2 + \frac{2}{\sqrt{d}} - \frac{8\sqrt{d}}{n}.$$

So,

$$1 + \kappa = \gamma \geq 2 + \frac{2}{\sqrt{d}} - \frac{8\sqrt{d}}{n} \Rightarrow \kappa \geq 1 + \frac{2}{\sqrt{d}} - \frac{8\sqrt{d}}{n}.$$

□

Lemma 31. Let $G = (V, E, w)$ be a connected weighted graph with n vertices and let d be the combinatorial degree of some vertex. Then

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{2}{\sqrt{d}} - \frac{8\sqrt{d}}{n}.$$

Proof. Take $\varepsilon := \frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} - 1 \geq 0$. By Lemma 27, there is $\alpha \in \mathbb{R}_{++}$ such that αG is a $(1+\varepsilon)$-spectral sparsifier of K_V. Note that each vertex has the same combinatorial degree in G and in αG. Hence, by Theorem 30,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} = 1 + \frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} - 1 = 1 + \varepsilon \geq 1 + \frac{2}{\sqrt{d}} - \frac{8\sqrt{d}}{n}. \qquad \square$$

Corollary 32 ([Srivastava and Trevisan, 2018, Claim 2.1])**.** Let $G = (V, E, w)$ be a connected weighted graph and suppose that $\delta(G) \leq d/4$ where $d := 2|E|/|V|$. Then

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - \frac{4\sqrt{d}}{n}.$$

Proof. By the hypothesis, there is a vertex in G with combinatorial degree $d' \leq d/4$. Hence, using Lemma 31,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{2}{\sqrt{d'}} - \frac{8\sqrt{d'}}{n} \geq 1 + \frac{2}{\sqrt{d/4}} - \frac{8\sqrt{d/4}}{n} = 1 + \frac{4}{\sqrt{d}} - \frac{4\sqrt{d}}{n}.$$

$\square$

Lemma 33 ([Srivastava and Trevisan, 2018, Claim 2.2])**.** Let $\varepsilon > 0$. Set $C_\varepsilon := \sqrt{16+\varepsilon}/(\sqrt{16+\varepsilon} - 4) > 0$. Let $G = (V, E, w)$ be a connected weighted graph on n vertices. Set $\alpha := (\Delta_w(G))^{-1}$. Suppose the average combinatorial degree d of G satisfies $d \geq 16 + \varepsilon$ and suppose that there exists $u \in V$ such that $\alpha w_u \leq 1 - 4/\sqrt{d}$. Then

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - C_\varepsilon \frac{1}{n}.$$

Proof. Let $u, v \in V$ such that $\alpha w_u \leq 1 - 4/\sqrt{d}$ and $w_v = 1/\alpha$. Define the function $f : V \to \mathbb{R}$ as

$$f(i) = \begin{cases} 1 & \text{if } i = u, \\ \frac{-1}{n-1} & \text{otherwise,} \end{cases} \qquad \text{for each } i \in V.$$

Note that

$$\mathbb{1}^T f = \sum_{i \in V} f(i) = 1 + \sum_{i \in V \setminus \{u\}} f(i) = 1 + \sum_{i \in V \setminus \{u\}} \frac{-1}{n-1} = 1 - \frac{n-1}{n-1} = 0,$$

hence $f \perp \mathbb{1}$, and so $\lambda_2^{\uparrow}(L_G) \leq \frac{f^T L_G f}{\|f\|^2}$. Also we have that

$$\|f\|^2 = \sum_{i\in V} f(i)^2 = 1 + \sum_{i\in V\setminus\{u\}} f(i)^2 = 1 + \sum_{i\in V\setminus\{u\}} \left(\frac{-1}{n-1}\right)^2 = 1 + \frac{n-1}{(n-1)^2} = 1 + \frac{1}{n-1}.$$

Since f is constant over $V \setminus \{u\}$,

$$\begin{aligned} f^T L_{\alpha G} f &= \alpha \sum_{ij\in E} w_{ij} \left(f(i) - f(j)\right)^2 = \sum_{i\in N(u)} \alpha w_{iu} \left(f(i) - f(u)\right)^2 \\ &= \alpha w_u \left(1 + \frac{1}{n-1}\right)^2 \leq \left(1 - \frac{4}{\sqrt{d}}\right)\left(1 + \frac{1}{n-1}\right)^2, \end{aligned}$$

where the last inequality uses the hypothesis $\alpha w_u \leq 1 - \frac{4}{\sqrt{d}}$. So,

$$\lambda_2^{\uparrow}(\alpha L_G) \leq \frac{f^T L_{\alpha G} f}{\|f\|^2} = \frac{f^T L_{\alpha G} f}{1 + \frac{1}{n-1}} \leq \left(1 - \frac{4}{\sqrt{d}}\right)\left(1 + \frac{1}{n-1}\right)^2 \frac{1}{1 + \frac{1}{n-1}} = \left(1 - \frac{4}{\sqrt{d}}\right)\left(1 + \frac{1}{n-1}\right).$$

Define the function $g : V \to \mathbb{R}$ as $g := e_v$. Note that $\|g\|^2 = 1$ and

$$g^T L_{\alpha G} g = e_v^T L_G e_v = \sum_{ij\in E} \alpha w_{ij}([i = v] - [j = v])^2 = \sum_{i\in N(v)} \alpha w_{iv} = 1.$$

So ,

$$\lambda_n^{\uparrow}(L_{\alpha G}) \geq \frac{g^T L_{\alpha G} g}{\|g\|^2} = 1.$$

Finally, we can bound the ratio

$$\begin{aligned} \frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} = \frac{\lambda_n^{\uparrow}(L_{\alpha G})}{\lambda_2^{\uparrow}(L_{\alpha G})} &\geq \frac{1}{\left(1 - \frac{4}{\sqrt{d}}\right)\left(1 + \frac{1}{n-1}\right)} = \frac{1}{\left(\frac{\sqrt{d}-4}{\sqrt{d}}\right)\left(\frac{n}{n-1}\right)} \\ &= \frac{\sqrt{d}n - \sqrt{d}}{\sqrt{d}n - 4n} = \frac{\sqrt{d}n}{\sqrt{d}n - 4n} - \frac{\sqrt{d}}{\sqrt{d}n - 4n} \\ &= 1 + \frac{4n}{\sqrt{d}n - 4n} - \frac{\sqrt{d}}{\sqrt{d}n - 4n} = 1 + \frac{4}{\sqrt{d} - 4} - \frac{\sqrt{d}}{\sqrt{d}n - 4n} \\ &\geq 1 + \frac{4}{\sqrt{d}} - \frac{\sqrt{d}}{\sqrt{d}n - 4n}. \end{aligned}$$

Note that $C_\varepsilon \geq \frac{\sqrt{d}}{\sqrt{d}-4}$. So,

$$C_\varepsilon \frac{1}{n} \geq \frac{\sqrt{d}}{\sqrt{d}n - 4n},$$

Whence,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - \frac{\sqrt{d}}{\sqrt{d}n - 4n} \geq 1 + \frac{4}{\sqrt{d}} - C_\varepsilon \frac{1}{n}.$$ □

Lemma 34 (see [Srivastava and Trevisan, 2018, Claim 2.3]). Let $\varepsilon > 0$. Let $d \geq 16 + \varepsilon$.

Set $C_\varepsilon := \sqrt{16+\varepsilon}/(\sqrt{16+\varepsilon}-4)$ and set $C_d := 1+4/\sqrt{d}$. Let $G=(V,E,w)$ be a connected weighted graph on n vertices such that d is the average combinatorial degree of G. Set $\alpha := (\Delta_w(G))^{-1}$. Suppose that there exists $e \in E$ such that $\alpha w_e > 8/\sqrt{d}$. Set $C_\varepsilon := \sqrt{16+\varepsilon}/(\sqrt{16+\varepsilon}-4)$ and set $C_d := 1+4/\sqrt{d}$. Then

$$\frac{\lambda_n^\uparrow(L_G)}{\lambda_2^\uparrow(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - \max\{C_\varepsilon, C_d\}\frac{1}{n}.$$

Proof. The maximum weighted degree of αG is 1, so $\mathrm{Tr}(L_{\alpha G}) \leq n$. By Lemma 15 the sum of the eigenvalues is less than or equal to n. Since $\lambda_1^\uparrow(L_{\alpha G}) = 0$, one has

$$\lambda_2^\uparrow(L_{\alpha G}) \leq \frac{n}{n-1}, \tag{2.14}$$

otherwise

$$\mathrm{Tr}(L_{\alpha G}) = \mathbb{1}^T\lambda^\uparrow(L_{\alpha G}) = \sum_{i=2}^{n} \lambda_i^\uparrow(L_{\alpha G}) > \sum_{i=2}^{n} \frac{n}{n-1} = n,$$

a contradiction.
Let $u, v \in V$ be such that $uv \in E$ and $\alpha w_{uv} > 8/\sqrt{d}$. Set $f := e_u - e_v$. By Lemma 33 we can assume that $\alpha w_i \geq 1 - 4/\sqrt{d}$ for each $i \in V$. So

$$\begin{aligned} f^T L_{\alpha G} f &= \alpha w_{uv}(f(u)-f(v))^2 + \sum_{i \in N(v)\setminus\{u\}} \alpha w_{iv}(f(i)-f(v))^2 + \sum_{i \in N(u)\setminus\{v\}} \alpha w_{iu}(f(i)-f(u))^2 \\ &= 4\alpha w_{uv} + \alpha w_v - \alpha w_{uv} + \alpha w_u - \alpha w_{uv} = 2\alpha w_{uv} + \alpha w_v + \alpha w_u \\ &\geq 2\frac{8}{\sqrt{d}} + 2\left(1 - \frac{4}{\sqrt{d}}\right) = 2 + \frac{8}{\sqrt{d}}. \end{aligned}$$

Hence,

$$\lambda_n^\uparrow(L_{\alpha G}) \geq \frac{f^T L_{\alpha G} f}{\|f\|^2} \geq \frac{2 + 8/\sqrt{d}}{2} = 1 + \frac{4}{\sqrt{d}}. \tag{2.15}$$

Now we can bound the ratio using (2.14) and (2.15):

$$\begin{aligned} \frac{\lambda_n^\uparrow(L_G)}{\lambda_2^\uparrow(L_G)} &= \frac{\lambda_n^\uparrow(L_{\alpha G})}{\lambda_2^\uparrow(L_{\alpha G})} \geq \frac{1+\frac{4}{\sqrt{d}}}{\frac{n}{n-1}} = \frac{(n-1)(\sqrt{d}+4)}{n\sqrt{d}} \\ &= \frac{n\sqrt{d} + 4n - \sqrt{d} - 4}{n\sqrt{d}} = 1 + \frac{4}{\sqrt{d}} - \frac{1}{n}\left(1 + \frac{4}{\sqrt{d}}\right) \\ &= 1 + \frac{4}{\sqrt{d}} - C_d\frac{1}{n}. \end{aligned}$$

□

The *ball* of radius $\ell \in \mathbb{Z}_+$ centered at $r \in V$ is defined as

$$B_\ell(r) := \{v \in V : \mathrm{dist}(v, r) \leq \ell\}.$$

Also, we denote the set of vertices at the *boundary* of the ball of radius $\ell \in \mathbb{Z}_+$ centered at $r \in V$ as

$$\mathrm{bd}(B_\ell(r)) := \{v \in V : \mathrm{dist}(v, r) = \ell\}.$$

Lemma 35. Let $G = (V, E, w)$ be a weighted graph. Set $\bar{\ell} := (\mathrm{girth}(G) - 1)/2$. Let $r \in V$. Let $\ell \in \mathbb{Z}_{++}$ such that $\ell \leq \bar{\ell}$. Set $T_{\ell-1} := G\big[B_{\ell-1}(r)\big]$. Then $T_{\ell-1}$ is a tree.

Proof. The proof is by induction. For $\ell = 1$, the subgraph T_0 is a tree because it has only a vertex and no edges. Let $\ell \leq \bar{\ell}$ and assume that $T_{\ell-2}$ is a tree. Suppose that there is a cycle C in $T_{\ell-1}$. Note that $T_{\ell-2}$ is a subgraph of $T_{\ell-1}$. Hence, by induction hypothesis, the cycle C must have at least one vertex $v \in V(T_{\ell-1}) \setminus V(T_{\ell-2})$. There exist two distinct vertices $u, w \in V(C)$ such that $u, w \in N(v)$, and there exist a path P_{ru} from r to u and a path P_{wr} from w to r, where both paths have length less than or equal to $\bar{\ell} - 1$. By concatenating P_{ru}, uv, vw, P_{wr} we form a closed trail. Since $u \neq w$, the closed trail formed contain a cycle with length less than or equal to the length of the closed trail, i.e., with length less than or equal to

$$\bar{\ell} - 1 + \bar{\ell} - 1 + 2 = 2\bar{\ell} = \mathrm{girth}(G) - 1 < \mathrm{girth}(G),$$

a contradiction. So $T_{\ell-1}$ is a tree. □

Lemma 36. Let $G = (V, E, w)$ be a weighted graph. Set $\bar{\ell} := (\mathrm{girth(G)} - 1)/2$. Let $r \in V$. Let $\ell \in \mathbb{Z}_{++}$ such that $\ell \leq \bar{\ell}$. Set $T_{\ell-1} := G\big[B_{\ell-1}(r)\big]$. Let $u, v \in \mathrm{bd}(B_{\ell-1}(r))$ be distinct. Then u and v have no commom neighbor in $V \setminus V(T_{\ell-1})$.

Proof. By Lemma 35, $T_{\ell-1}$ is a tree. Take $T_{\ell-1}$ to be rooted at r. Hence $\mathrm{bd}(B_{\ell-1}(r))$ is the set of leaves of $T_{\ell-1}$. Suppose that there are leaves u, v of $T_{\ell-1}$ that are adjacent to some $x \in V \setminus B_{\ell-1}(r)$. Then there exist a path P_{ru} with length less than or equal to $\bar{\ell} - 1$, and a path P_{vr} with length less than or equal to $\bar{\ell} - 1$. So we can build a cycle by concatenating P_{ru}, ux, xv and P_{vr}, with length at most

$$\bar{\ell} - 1 + \bar{\ell} - 1 + 2 = 2\bar{\ell} = 2(\mathrm{girth}(G) - 1)/2 = \mathrm{girth}(G) - 1 < \mathrm{girth}(G),$$

a contradiction. □

Lemma 37 (see [Srivastava and Trevisan, 2018, Claim 2.4])**.** Let $G = (V, E, w)$ be a weighted graph of average combinatorial degree $d \geq 12$. Suppose that $\delta(G) \geq d/4$. Set $\bar{\ell} := (\mathrm{girth}(G) - 1)/2$. Let $r \in V$. Then for every positive integer $\ell \leq \bar{\ell}$, we have that

$$|B_\ell(r)| \leq \frac{2n}{\left(\frac{d}{4} - 1\right)^{\bar{\ell}-\ell}}.$$

Proof. Let $\ell \in \mathbb{Z}_{++}$ such that $\ell \leq \bar{\ell}$. Set $T_{\ell-1} := G\big[B_{\ell-1}(r)\big]$. By Lemma 35, $T_{\ell-1}$ is a tree. Take $T_{\ell-1}$ to be rooted at r. Since $\delta(G) \geq d/4 \geq 3$, each internal node has at least 2 children, so the number of internal nodes is less than or equal to the number of leaves, which will be the vertices at distance $\ell - 1$ from r. Let β be the number of leaves of $T_{\ell-1}$. So $|B_{\ell-1}(r)| \leq 2\beta$. Since $\delta(G) \geq 3$, each leaf of $T_{\ell-1}$ has at least 2 neighbors in the set $V \setminus B_{\ell-1}(r)$, and by Lemma 36, there is no commom neighbor outside $T_{\ell-1}$ between two distinct leaves. Consider the set $\mathrm{bd}(B_\ell(r))$ that has the vertices outside of $T_{\ell-1}$ that are adjacent to some leaf of $T_{\ell-1}$. Hence, one has $|\mathrm{bd}(B_\ell(r))| \geq 2\beta \geq |B_{\ell-1}(r)|$. So,

$$|B_\ell(r)| = |B_{\ell-1}(r)| + \mathrm{bd}(B_\ell(r)) \leq 2|\mathrm{bd}(B_\ell(r))|.$$

Hence, it is sufficient to prove that

$$|\mathrm{bd}(B_\ell(r))| \leq \frac{n}{\left(\frac{d}{4}-1\right)^{\bar\ell-\ell}}.$$

Set $T_{\bar\ell-1} := G\big[B_{\bar\ell-1}(r)\big]$. By Lemma 35, the graph $T_{\bar\ell-1}$ is a tree. Take $T_{\bar\ell-1}$ to be rooted at r. Note that each internal node has at least $\delta(G)-1 \geq d/4-1$ children. Also, each leaf of $T_{\bar\ell-1}$ has at least $\delta(G)-1 \geq d/4-1$ neighbors outside of $T_{\bar\ell-1}$, and by Lemma 36, there is no commom neighbor outside of $T_{\bar\ell-1}$ between two distinct leaves. Denote the number of vertices at distance $i \leq \bar\ell$ from r as t_i. So for each positive integer $i \leq \bar\ell$, we have that $t_{i-1}(d/4-1) \leq t_i$. More generally, for each positive integer $i \leq \bar\ell$ and for each nonnegative integer $\beta \leq i$, we have that $t_{i-\beta}(d/4-1)^\beta \leq t_i$. Clearly $t_{\bar\ell} \leq n$. Hence,

$$t_\ell\left(\frac{d}{4}-1\right)^{\bar\ell-\ell} \leq t_{\bar\ell} \leq n.$$

So,

$$|\mathrm{bd}(B_\ell(r))| = t_\ell \leq \frac{n}{\left(\frac{d}{4}-1\right)^{\bar\ell-\ell}}.$$ □

2.3 Main Result

Throughout this section we consider the following hypotheses. Let $G=(V,E,w)$ be a connected weighted graph of average combinatorial degree $d := 2|E|/|V|$ such that

$$\Delta_w(G) = 1, \tag{2.16a}$$
$$\delta(G) \geq d/4, \tag{2.16b}$$
$$\delta_w(G) > 1 - 4/\sqrt{d}, \tag{2.16c}$$
$$w_e \leq 8/\sqrt{d} \text{ for each } e \in E. \tag{2.16d}$$

Let $k \in \mathbb{Z}_{++}$ be such that $k < (\mathrm{girth}(G)-1)/2$ and define the function $f_r : V \to \mathbb{R}$ for each $r \in V$ as

$$f_r(v) = \begin{cases} 0 & \text{if } \mathrm{dist}(r,v) > k, \\ 1 & \text{if } r = v, \\ \sqrt{\prod_{e\in E(P_{rv})} w_e} & \text{otherwise, where } P_{rv} \text{ is the unique path between u and v in } G. \end{cases} \tag{2.17}$$

Lemma 38 (see [Srivastava and Trevisan, 2018, Equation (3)])**.** Let $G=(V,E,w)$ be a connected weighted graph of average combinatorial degree $d := 2|E|/|V| \geq 144$ such that (2.16) holds. Let $k \in \mathbb{Z}_{++}$ such that $k < (\mathrm{girth}(G)-1)/2$. Let $r \in V$ and define $f_r : V \to \mathbb{R}$ as in (2.17). Then

$$\left(1-\frac{12}{\sqrt{d}}\right)^k (k+1) \leq \|f_r\|^2 \leq k+1 \quad \text{for each } v \in V.$$

Proof. Let T be the subgraph induced by $B_k(r)$. By Lemma 35, the graph T is a tree. Take T to be rooted at r. Denote the parent of each vertex $v \in V(T) \setminus \{r\}$ as $p(v)$. Note that

$$f_r(v)^2 = \prod_{e \in E(P_{rv})} w_e = w_{vp(v)} \prod_{e \in E(P_{rp(v)})} w_e = w_{vp(v)} f_r(p(v))^2 \quad \text{for each } v \in V(T) \setminus \{r\}.$$

Hence, for each $0 \le \ell \le k-1$, one has

$$\left\| f_r \restriction_{\mathrm{bd}(B_{\ell+1}(r))} \right\|^2 = \sum_{v \in \mathrm{bd}(B_{\ell+1}(r))} f_r(v)^2 = \sum_{v \in \mathrm{bd}(B_{\ell+1}(r))} w_{vp(v)} f_r(p(v))^2 = \sum_{v \in \mathrm{bd}(B_\ell(r))} f_r(v)^2 (w_v - w_{vp(v)}).$$

By (2.16a) and since $w_e \ge 0$ for each $e \in E$, we have that $w_v - w_{vp(v)} \le 1 - 0 = 1$ for each $v \in V$. So,

$$\left\| f_r \restriction_{\mathrm{bd}(B_{\ell+1}(r))} \right\|^2 = \sum_{v \in \mathrm{bd}(B_\ell(r))} f_r(v)^2 (w_v - w_{vp(v)}) \le \sum_{v \in \mathrm{bd}(B_\ell(r))} f_r(v)^2 = \left\| f_r \restriction_{\mathrm{bd}(B_\ell(r))} \right\|^2. \tag{2.18}$$

By (2.16c) and (2.16d),

$$w_v - w_{vp(v)} \ge 1 - \frac{4}{\sqrt{d}} - \frac{8}{\sqrt{d}} = 1 - \frac{12}{\sqrt{d}}.$$

So

$$\begin{aligned}
\left\| f_r \restriction_{\mathrm{bd}(B_{\ell+1}(r))} \right\|^2 &= \sum_{v \in \mathrm{bd}(B_\ell(r))} f_r(v)^2 (w_v - w_{vp(v)}) \\
&\ge \left(1 - \frac{12}{\sqrt{d}}\right) \sum_{v \in \mathrm{bd}(B_\ell(r))} f_r(v)^2 \\
&= \left(1 - \frac{12}{\sqrt{d}}\right) \left\| f_r \restriction_{\mathrm{bd}(B_\ell(r))} \right\|^2.
\end{aligned} \tag{2.19}$$

Note that $\left\| f_r \restriction_{\mathrm{bd}(B_0(r))} \right\|^2 = 1$. By (2.19),

$$\left\| f_r \restriction_{\mathrm{bd}(B_i(r))} \right\|^2 \le \left\| f_r \restriction_{\mathrm{bd}(B_0(r))} \right\|^2 = 1 \quad \text{for each } 1 \le i \le k.$$

So,

$$\|f_r\|^2 = \sum_{i=0}^{k} \left\| f_r \restriction_{\mathrm{bd}(B_i(r))} \right\|^2 \le \sum_{i=0}^{k} 1 = k+1.$$

By (2.18),

$$\left(1 - \frac{12}{\sqrt{d}}\right)^k = \left\| f_r \restriction_{\mathrm{bd}(B_0(r))} \right\|^2 \left(1 - \frac{12}{\sqrt{d}}\right)^k \le \left\| f_r \restriction_{\mathrm{bd}(B_k(r))} \right\|^2.$$

Since $\left\| f_r \restriction_{\mathrm{bd}(B_k(r))} \right\|^2 \le \left\| f_r \restriction_{\mathrm{bd}(B_i(r))} \right\|^2$ for each $0 \le i \le k$, we have that

$$\|f_r\|^2 = \sum_{i=0}^{k} \left\| f_r \restriction_{\mathrm{bd}(B_i(r))} \right\|^2 \ge \sum_{i=0}^{k} \left\| f_r \restriction_{\mathrm{bd}(B_k(r))} \right\|^2 \ge \sum_{i=0}^{k} \left(1 - \frac{12}{\sqrt{d}}\right)^k = (k+1)\left(1 - \frac{12}{\sqrt{d}}\right)^k. \qquad \square$$

Lemma 39 (see [Srivastava and Trevisan, 2018, Equation (4)]). Let $G = (V, E, w)$ be a connected weighted graph of average combinatorial degree $d := 2|E|/|V| \geq 144$ such that (2.16) holds. Set $k := \lfloor d^{1/8} \rfloor$. Suppose that $\mathrm{girth}(G) \geq 2d^{1/8} + 5$. Let $r \in V$ and define $f_r : V \to \mathbb{R}$ as in (2.17). Then

$$\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r\|^2 \geq \|f_r\|^2\left(1 - \frac{50}{d^2}\right).$$

Proof. Since $\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} = \mathbb{1}\mathbb{1}^T/\mathbb{1}^T\mathbb{1}$, we have that

$$\begin{aligned}\|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r\|^2 &= \left(\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r\right)^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r = f_r^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}}^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r \\ &= f_r^T \mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r = f_r^T \frac{\mathbb{1}\mathbb{1}^T}{\mathbb{1}^T\mathbb{1}} f_r = \frac{(f_r^T\mathbb{1})^2}{n}.\end{aligned}$$

Furthermore, one has that $f_r(i) = \|f_r \restriction_{\{i\}}\| \leq \|f_r\|$ for each $i \in V$. So,

$$\frac{(f_r^T\mathbb{1})^2}{n} = \frac{1}{n}\Big(\sum_{i \in V} f_r(i)\Big)^2 = \frac{1}{n}\Big(\sum_{i \in \mathrm{supp}(f_r)} f_r(i)\Big)^2 = \frac{1}{n}\left(\mathbb{1}_{\mathrm{supp}(f_r)}^T f_r\right)^2.$$

By Cauchy-Schwarz inequality,

$$\mathbb{1}_{\mathrm{supp}(f_r)}^T f_r \leq \|f_r\| \|\mathbb{1}_{\mathrm{supp}(f_r)}^T\| = \|f_r\| |\mathrm{supp}(f_r)|^{1/2},$$

which implies that

$$\left(\mathbb{1}_{\mathrm{supp}(f_r)}^T f_r\right)^2 \leq \|f_r\|^2 |\mathrm{supp}(f_r)|.$$

Hence,

$$\frac{(f_r^T\mathbb{1})^2}{n} = \frac{1}{n}\left(\mathbb{1}_{\mathrm{supp}(f_r)}^T f_r\right)^2 \leq \frac{1}{n}\|f_r\|^2|\mathrm{supp}(f_r)|.$$

Since $\mathrm{girth}(G) \geq 2d^{1/8} + 5 \geq 2d^{1/8} + 1$ and $k \leq d^{1/8}$, we have that $\mathrm{girth}(G) \geq 2k + 1$, which implies that $k \leq (\mathrm{girth}(G) - 1)/2$. Note that, for each $v \in V$, one has that $f_r(v) \neq 0$ if and only if $v \in B_k(r)$. Hence $B_k(r) = \mathrm{supp}(f_r)$. By Lemma 37,

$$|\mathrm{supp}(f_r)| \leq \frac{2n}{\left(\frac{d}{4} - 1\right)^{\frac{\mathrm{girth}(G)-1}{2} - k}}.$$

By hypotheses, we have that $\mathrm{girth}(G) \geq 2k + 5$. Hence,

$$\begin{aligned}&\mathrm{girth}(G) \geq 2k + 5 = 2(k+2) + 1 \\ \Rightarrow &\mathrm{girth}(G) - 1 \geq 2(k+2) \\ \Rightarrow &\frac{\mathrm{girth}(G) - 1}{2} \geq 2.\end{aligned}$$

So,

$$|\mathrm{supp}(f_r)| \leq \frac{2n}{\left(\frac{d}{4} - 1\right)^{\frac{\mathrm{girth}(G)-1}{2} - k}} \leq \frac{2n}{\left(\frac{d}{4} - 1\right)^2}.$$

For $d \geq 20$, we have that $d/4 - 1 \geq d/5$. So,

$$|\mathrm{supp}(f_r)| \leq \frac{2n}{\left(\frac{d}{4} - 1\right)^2} \leq \frac{50n}{d^2}.$$

Hence,

$$\|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r\|^2 = \frac{(f_r^T \mathbb{1})^2}{n} \leq \frac{1}{n}\|f_r\|^2 |\mathrm{supp}(f_r)| \leq \frac{1}{n}\|f_r\|^2 \frac{50n}{d^2} = \|f_r\|^2 \frac{50}{d^2}.$$

By Lemma 17, we have that $\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r\|^2 = \|f_r\|^2 - \|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r\|^2$. So,

$$\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r\|^2 = \|f_r\|^2 - \|\mathrm{Proj}_{\mathrm{span}\{\mathbb{1}\}} f_r\|^2 \geq \|f_r\|^2 - \|f_r\|^2 \frac{50}{d^2} = \|f_r\|^2 \left(1 - \frac{50}{d^2}\right). \qquad \square$$

Consider a random walk $\langle X_0, \dots, X_k \rangle$ of k-steps on the graph G, we denote the event of moving from a vertex u to a vertex v as $X_x X_{x+1} = uv$. Note that this random walk considered induces a tree, so the walk backtracks at step $i \in [k-1]$ if $X_{i-1}X_i = X_i X_{i+1}$. Denote the event of the walk backtracking at step $i \in [k-1]$ as backtrack(i). Define $\pi : V \to R$ as

$$\pi(r) = \frac{w_r}{2\mathbb{1}^T w} \quad \text{for each } r \in V.$$

The above function is the stationary distribution, so if the distribution of X_0 is π and $\Pr(X_x X_{x+1} = uv) = w_{uv}/w_u$, for each $uv \in E$,

$$\Pr(X_i = r) = \pi(r) \quad \text{for each } i \in \{0, \dots, k\} \text{ and for each } v \in V.$$

For the following propositions we also will consider the following hypothesis. Let $\langle X_0, \dots, X_k \rangle$ be a random walk such that

$$\Pr(X_x X_{x+1} = uv) = \frac{w_{uv}}{w_u} \quad \text{for each } uv \in E, \tag{2.20a}$$

$$\text{the distribution of } X_0 \text{ is } \pi. \tag{2.20b}$$

Proposition 40 (see [Srivastava and Trevisan, 2018, Proposition 3.2]). Let $G = (V, E, w)$ be a connected weighted graph of average combinatorial degree $d := 2|E|/|V| \geq 16$ such that (2.16) holds. Let $k \in \mathbb{Z}_{++}$ such that $k < (\mathrm{girth}(G) - 1)/2$. Let $\langle X_0, \dots, X_k \rangle$ be a random walk such that (2.20) holds. Then

$$\mathbb{E}\left(\sum_{i=1}^{k} \sqrt{w(X_{i-1}, X_i)}\right) \geq \frac{k}{\sqrt{d}} - \frac{2k}{d}.$$

Proof. By (2.20),

$$\Pr(X_i = u) = \Pr(X_0 = u) = \pi(u) \quad \text{for each } u \in V \text{ and for each } i \in \{0, \dots, k\}.$$

So,

$$\begin{aligned}
\mathbb{E}\Bigg(\sum_{i\in[k]}\sqrt{w(X_{i-1},X_i)}\Bigg) &= \sum_{i\in[k]}\sum_{uv\in E}\sqrt{w_{uv}}\Pr(X_{i-1}X_i = uv)\\
&= \sum_{i\in[k]}\sum_{uv\in E}\sqrt{w_{uv}}\big(\Pr(X_{i-1}=u, X_i=v)+\Pr(X_{i-1}=v, X_i=u)\big)\\
&= \sum_{i\in[k]}\sum_{uv\in E}\sqrt{w_{uv}}\big(\pi(u)\Pr(X_xX_{x+1}=uv)+\pi(v)\Pr(X_xX_{x+1}=vu)\big)\\
&= k\sum_{uv\in E}\sqrt{w_{uv}}\Big(\frac{w_u}{2\mathbb{1}^Tw}\frac{w_{uv}}{w_u}+\frac{w_v}{2\mathbb{1}^Tw}\frac{w_{uv}}{w_v}\Big)\\
&= k\sum_{uv\in E}\sqrt{w_{uv}}\Big(\frac{2w_{uv}}{2\mathbb{1}^Tw}\Big) = k\frac{\sum_{uv\in E} w_{uv}^{3/2}}{\mathbb{1}^Tw}.
\end{aligned}$$

Note that $x^{3/2}$ is a convex function. So $\sum_{uv\in E} w_{uv}^{3/2}$ is minimized, while maintaining the sum $\sum_{uv\in E} w_{uv}$ constant, when all the edges has the same weight, i.e., when

$$\overline{w_e} := \frac{\mathbb{1}^Tw}{dn/2} \quad \text{for each } e\in E.$$

So,

$$\sum_{uv\in E} w_{uv}^{3/2} \ge \sum_{uv\in E}\Big(\frac{\mathbb{1}^Tw}{dn/2}\Big)^{3/2} = \frac{dn}{2}\Big(\frac{\mathbb{1}^Tw}{dn/2}\Big)^{3/2} = \frac{(\mathbb{1}^Tw)^{3/2}}{(dn/2)^{1/2}}. \tag{2.21}$$

By (2.16c),

$$\mathbb{1}^Tw = \frac{1}{2}\sum_{v\in V} w_v \ge \frac{1}{2}\sum_{v\in V}\Big(1-\frac{4}{\sqrt{d}}\Big) = \frac{n}{2}\Big(1-\frac{4}{\sqrt{d}}\Big). \tag{2.22}$$

From (2.22),

$$\frac{2\mathbb{1}^Tw}{n} \ge \Big(1-\frac{4}{\sqrt{d}}\Big). \tag{2.23}$$

Hence by (2.21) and (2.23),

$$\frac{\sum_{uv\in E} w_{uv}^{3/2}}{\mathbb{1}^Tw} \ge \frac{1}{\mathbb{1}^Tw}\frac{(\mathbb{1}^Tw)^{3/2}}{(dn/2)^{1/2}} = \Big(\frac{\mathbb{1}^Tw}{dn/2}\Big)^{1/2} = \Big(\frac{2\mathbb{1}^Tw}{dn}\Big)^{1/2} \ge \Big(1-\frac{4}{\sqrt{d}}\Big)^{1/2}\frac{1}{\sqrt{d}}.$$

Since $0 \le (1-4/\sqrt{d}) \le 1$, we have that $(1-4/\sqrt{d})^{1/2} \ge 1-4/\sqrt{d}$. So,

$$\frac{\sum_{uv\in E} w_{uv}^{3/2}}{\mathbb{1}^Tw} \ge \Big(1-\frac{4}{\sqrt{d}}\Big)^{1/2}\frac{1}{\sqrt{d}} \ge \Big(1-\frac{4}{\sqrt{d}}\Big)\frac{1}{\sqrt{d}} = \frac{1}{\sqrt{d}}-\frac{4}{d}.$$

Finally,

$$\mathbb{E}\Bigg(\sum_{i\in[k]}\sqrt{w(X_{i-1},X_i)}\Bigg) = k\frac{\sum_{uv\in E} w_{uv}^{3/2}}{\mathbb{1}^Tw} \ge k\Big(\frac{1}{\sqrt{d}}-\frac{4}{d}\Big) = \frac{k}{\sqrt{d}}-\frac{4k}{d}. \qquad \square$$

Proposition 41 (see [Srivastava and Trevisan, 2018, Proposition 3.3]). Let $G = (V, E, w)$

be a connected weighted graph of average combinatorial degree $d := 2|E|/|V| \geq 25$ such that (2.16) holds. Let $k \in \mathbb{Z}_{++}$ such that $k < (\mathrm{girth}(G) - 1)/2$. Let $\langle X_0, \dots, X_k \rangle$ be a random walk such that (2.20) holds. Then

$$\mathbb{E}\left([\text{the walk backtracks}] \sum_{i=1}^{k} \sqrt{w(X_{i-1}, X_i)} \right) \leq \frac{80\sqrt{2}k^2}{d^{3/4}}.$$

Proof. By (2.16d),

$$\sum_{i=1}^{k} \sqrt{w(X_{i-1}, X_i)} \leq \sum_{i=1}^{k} \Big(\frac{8}{\sqrt{d}}\Big)^{1/2} = \frac{2\sqrt{2}k}{d^{1/4}}.$$

Hence,

$$\begin{aligned}
\mathbb{E}\left([\text{the walk backtracks}] \sum_{i=1}^{k} \sqrt{w(X_{i-1}, X_i)} \right) &\leq \mathbb{E}\left([\text{the walk backtracks}] \frac{2\sqrt{2}k}{d^{1/4}} \right) \\
&= \frac{2\sqrt{2}k}{d^{1/4}} \mathbb{E}\big([\text{the walk backtracks}]\big).
\end{aligned}$$

For all $i \in [k-1]$, the probability of the event $X_{i-1}X_i = X_i X_{i+1}$ is the probability of going from the vertex X_i to the vertex X_{i-1}. So,

$$\Pr(\mathrm{backtrack}(i)) = \Pr(X_x X_{x+1} = X_i X_{i-1}) = \frac{w_{X_i X_{i-1}}}{w_{X_i}}.$$

By (2.16d) and (2.16c),

$$\Pr(\mathrm{backtrack}(i)) = \frac{w_{X_i X_{i-1}}}{w_{X_i}} \leq \frac{8/\sqrt{d}}{1 - 4/\sqrt{d}}.$$

The event of the walk backtracking is equal to the event $\bigcup_{i \in [k-1]} \mathrm{backtrack}(i)$. Hence,

$$\begin{aligned}
\Pr(\text{the walk backtracks}) &= \Pr\left(\bigcup_{i \in [k-1]} \mathrm{backtrack}(i) \right) \leq \sum_{i=1}^{k-1} \Pr(\mathrm{backtrack}(i)) \\
&\leq \sum_{i=1}^{k-1} \frac{8/\sqrt{d}}{1 - 4/\sqrt{d}} = (k-1)\frac{8/\sqrt{d}}{1 - 4/\sqrt{d}} = (k-1)\frac{8}{\sqrt{d} - 4} \\
&\leq \frac{8k}{\sqrt{d} - 4} = \frac{40k}{\sqrt{d}} + \frac{160k - 32k\sqrt{d}}{\sqrt{d}(\sqrt{d} - 4)}.
\end{aligned}$$

Since $d \geq 25$ and $k \geq 0$, we have that $160k - 32k\sqrt{d} \leq 160k - 160k = 0$ and $\sqrt{d}\,(\sqrt{d} - 4) \geq 5(5-4) = 5 \geq 0$. So,

$$\Pr(\text{the walk backtracks}) \leq \frac{40k}{\sqrt{d}} + \frac{160k - 32k\sqrt{d}}{\sqrt{d}(\sqrt{d} - 4)} \leq \frac{40k}{\sqrt{d}}.$$

Since [the walk backtracks] $\in \{0, 1\}$,

$$\mathbb{E}\big([\text{the walk backtracks}]\big) \;=\; \Pr\big([\text{the walk backtracks}]\big).$$

So,

$$\begin{aligned}\mathbb{E}\left([\text{the walk backtracks}] \sum_{i=1}^{k} \sqrt{w(X_{i-1}, X_i)}\right) &\le \frac{2\sqrt{2}k}{d^{1/4}} \mathbb{E}\big([\text{the walk backtracks}]\big) \\ &= \frac{2\sqrt{2}k}{d^{1/4}} \Pr([\text{the walk backtracks}]) \\ &\le \frac{2\sqrt{2}k}{d^{1/4}} \frac{40k}{\sqrt{d}} = \frac{80\sqrt{2}k^2}{d^{3/4}}. \qquad \square\end{aligned}$$

Lemma 42 (see [Srivastava and Trevisan, 2018, Lemma 3.1])**.** Let $d \ge 144$. Set $k := d^{1/8}$. There is $\gamma \in \mathbb{R}_{++}$ satisfying the following. If $G = (V, E, w)$ is a connected weighted graph of average combinatorial degree $d = 2|E|/|V|$ such that (2.16) holds. Suppose that $\operatorname{girth}(G) \ge 2d^{1/8} + 5$. Define $f_r : V \to \mathbb{R}$ as in (2.17) for each $r \in V$. Then there is a vertex $r \in V$ such that

$$f_r^T A_G f_r \ge \frac{2k}{\sqrt{d}} - \gamma \frac{k^2}{d^{3/4}}.$$

Proof. Set $\gamma := \gamma_1(12 + 160\sqrt{2})$, where

$$\gamma_1 := \sum_{i=0}^{3} \frac{e^i}{i^i} \frac{1}{2^{5i/8}} + \frac{2}{2 - 2^{3/8}}.$$

Let $r \in V$. Set $T := G[B_k(r)]$. Since $k \le d^{1/8}$ and $\operatorname{girth}(G) \ge 2d^{1/8} + 5$, we have that $k < (\operatorname{girth}(G) - 1)/2$. By Lemma 35, the graph T is a tree. Take T to be rooted at r. Denote the parent of each vertex $v \in V(T) \setminus \{r\}$ in T as $p(v)$. Since T is a tree rooted at r we have that $E(T) = \bigcup_{v \in V(T) \setminus \{r\}} p(v)v$. By (2.17), we have that $f_r(v) = \sqrt{w_{p(v)v}} f_r(p(v))$, also $f_r(v) = 0$ for each $v \in V \setminus V(T)$. So,

$$f_r^T A_G f_r = 2 \sum_{uv \in E(T)} w_{uv} f_r(u) f_r(v) = 2 \sum_{v \in V(T) \setminus \{r\}} w_{p(v)v} f_r(p(v)) f_r(v) = 2 \sum_{v \in V(T) \setminus \{r\}} \sqrt{w_{p(v)v}} f_r(v)^2.$$

Let $\langle X_0, \dots, X_k \rangle$ be a random walk such that (2.20) holds. By (2.16c) and (2.20a),

$$w_{uv} = w_u \Pr(X_x X_{x+1} = uv) \ge \left(1 - \frac{4}{\sqrt{d}}\right) \Pr(X_x X_{x+1} = uv). \tag{2.24}$$

Let $v \in V$ be a vertex such that $\ell := \operatorname{dist}(r, v) \le k$. Note that $v \in V(T)$. So there is a unique path between r and v in G, in particular in T, denote this path as $P := \langle r = v_0, \dots, v_\ell = v \rangle$. Using (2.24),

$$f_r(v)^2 = \prod_{i \in [\ell]} w_{v_{i-1}v_i} \ge \prod_{i \in [\ell]} \left(1 - \frac{4}{\sqrt{d}}\right) \Pr(X_x X_{x+1} = v_{i-1}v_i) = \left(1 - \frac{4}{\sqrt{d}}\right)^{\ell} \Pr(X_\ell = v),$$

where in the last equality we are using the fact that P is the unique path between r and v. Hence, to reach v in ℓ steps one has to choose exactly the edges of P. Note that the event $X_\ell = v$ is either 1, when the event happen, or 0, when it does not happen, so $\Pr(X_\ell = v) = \mathbb{E}\big([X_\ell = v]\big)$. Hence,

$$\begin{aligned}
f_r^T A_G f_r &= 2 \sum_{v \in V(T)\setminus\{r\}} \sqrt{w_{p(v)v}} f_r(v)^2 \\
&\geq 2 \sum_{v \in V(T)\setminus\{r\}} \sqrt{w_{p(v)v}} \left(1 - \frac{4}{\sqrt{d}}\right)^{\operatorname{dist}(r,v)} \Pr(X_{\operatorname{dist}(r,v)} = v) \\
&\geq 2 \sum_{v \in V(T)\setminus\{r\}} \sqrt{w_{p(v)v}} \left(1 - \frac{4}{\sqrt{d}}\right)^{k} \Pr(X_{\operatorname{dist}(r,v)} = v) \\
&= 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \sum_{v \in V(T)\setminus\{r\}} \sqrt{w_{p(v)v}} \mathbb{E}_r\big([X_{\operatorname{dist}(r,v)} = v]\big) \\
&= 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \mathbb{E}_r\left(\sum_{v \in V(T)\setminus\{r\}} \sqrt{w_{p(v)v}} [X_{\operatorname{dist}(r,v)} = v]\right).
\end{aligned}$$

Let $i \in [k]$ and let $u \in V(T)$ such that $\operatorname{dist}(r, u) = i$. Note that $X_i = u$ if and only if $\operatorname{dist}(r, X_i) = i$. So, we have that $[X_i = u] = [\operatorname{dist}(r, X_i) = i]$. Also, for every $j \in [k]$ there is only one vertex $z \in V(T)$ such that $X_i = z$. Hence,

$$\begin{aligned}
f_r^T A_G f_r &\geq 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \mathbb{E}_r\left(\sum_{v \in V(T)\setminus\{r\}} \sqrt{w_{p(v)v}} [X_{\operatorname{dist}(r,v)} = v]\right) \\
&= 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \mathbb{E}_r\left(\sum_{i \in [k]} \sqrt{w_{X_{i-1}X_i}} [\operatorname{dist}(r, X_i) = i]\right)
\end{aligned}$$

Note that if the event $\operatorname{dist}(r, X_i) = i$ happens, then the walk does not backtrack up to step i. Also, the probability of the walk not backtracking is smaller than the probability of the walk not backtracking up to the step i for $i \in [k]$. So,

$$\begin{aligned}
f_r^T A_G f_r &\geq 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \mathbb{E}_r\left(\sum_{i \in [k]} \sqrt{w_{X_{i-1}X_i}} [\operatorname{dist}(r, X_i) = i]\right) \\
&= 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \mathbb{E}_r\left(\sum_{i \in [k]} \sqrt{w_{X_{i-1}X_i}} [\text{the walk does not backtracks up to step } i]\right) \\
&\geq 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \mathbb{E}_r\left(\sum_{i \in [k]} \sqrt{w_{X_{i-1}X_i}} [\text{the walk does not backtracks up to step } k]\right) \\
&\geq 2\left(1 - \frac{4}{\sqrt{d}}\right)^{k} \mathbb{E}_r\left([\text{the walk does not backtracks}] \sum_{i \in [k]} \sqrt{w_{X_{i-1}X_i}}\right).
\end{aligned}$$

Using the complementary event of *the walk does not backtracks*,

$$\mathbb{E}_r\big([\text{the walk does not backtracks}]\big) = 1 - \mathbb{E}_r\big([\text{the walk backtracks}]\big).$$

Hence,

$$
\begin{aligned}
f_r^T A_G f_r &\geq 2\left(1-\frac{4}{\sqrt{d}}\right)^k \mathbb{E}_r\left([\text{the walk does not backtracks}] \sum_{i\in[k]} \sqrt{w_{X_{i-1}X_i}}\right) \\
&= 2\left(1-\frac{4}{\sqrt{d}}\right)^k \left(\mathbb{E}_r\left(\sum_{i\in[k]} \sqrt{w_{X_{i-1}X_i}}\right) - \mathbb{E}_r\left([\text{the walk backtracks}] \sum_{i\in[k]} \sqrt{w_{X_{i-1}X_i}}\right)\right).
\end{aligned}
$$

Using (2.20b), we can average over all possible roots. By Proposition 40 and Proposition 41,

$$
\sum_{r\in V} \pi(r) f_r^T A_G f_r \geq 2\left(1-\frac{4}{\sqrt{d}}\right)^k \left(\frac{k}{\sqrt{d}} - \frac{2k}{d} - \frac{80\sqrt{2}k^2}{d^{3/4}}\right)
$$

Define the sets

$$
\text{Even} := \{i \in \{0,\dots,k\} : i \text{ is even}\} \quad \text{and} \quad \text{Odd} := \{i \in \{0,\dots,k\} : i \text{ is odd}\}.
$$

Expanding the expression,

$$
\begin{aligned}
&2\left(1-\frac{4}{\sqrt{d}}\right)^k \left(\frac{k}{\sqrt{d}} - \frac{2k}{d} - \frac{80\sqrt{2}k^2}{d^{3/4}}\right) \\
&= 2\frac{k}{\sqrt{d}} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{4^i}{d^{i/2}} - 2\frac{2k}{d} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{4^i}{d^{i/2}} - 2\frac{80\sqrt{2}k^2}{d^{3/4}} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{4^i}{d^{i/2}} \\
&\geq \frac{2k}{\sqrt{d}} - \frac{2k}{\sqrt{d}} \sum_{i\in\text{Odd}} \binom{k}{i} \frac{4^i}{d^{i/2}} - \frac{4k}{d} \sum_{i\in\text{Even}} \binom{k}{i} \frac{4^i}{d^{i/2}} - \frac{160\sqrt{2}k^2}{d^{3/4}} \sum_{i\in\text{Even}} \binom{k}{i} \frac{4^i}{d^{i/2}} \\
&= \frac{2k}{\sqrt{d}} - \frac{2k}{\sqrt{d}} \frac{4}{d^{1/2}} \sum_{i\in\text{Even}\setminus\{k\}} \binom{k-1}{i} \frac{4^i}{d^{i/2}} - \frac{4k}{d} \sum_{i\in\text{Even}} \binom{k}{i} \frac{4^i}{d^{i/2}} - \frac{160\sqrt{2}k^2}{d^{3/4}} \sum_{i\in\text{Even}} \binom{k}{i} \frac{4^i}{d^{i/2}} \\
&= \frac{2k}{\sqrt{d}} - \frac{8k}{d} \sum_{i\in\text{Even}\setminus\{k\}} \binom{k-1}{i} \frac{4^i}{d^{i/2}} - \frac{4k}{d} \sum_{i\in\text{Even}} \binom{k}{i} \frac{4^i}{d^{i/2}} - \frac{160\sqrt{2}k^2}{d^{3/4}} \sum_{i\in\text{Even}} \binom{k}{i} \frac{4^i}{d^{i/2}}.
\end{aligned} \tag{2.25}
$$

Note that

$$
\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{2^{2i}}{d^{i/2}} \leq \sum_{i=0}^{k} \binom{k}{i} \frac{2^{2i}}{d^{i/2}} \leq \sum_{i=0}^{k} \frac{e^i k^i}{i^i} \frac{2^{2i}}{d^{i/2}} = \sum_{i=0}^{k} \frac{e^i}{i^i} d^{i/8} \frac{2^{2i}}{d^{i/2}} = \sum_{i=0}^{k} \frac{e^i}{i^i} \frac{2^{2i}}{d^{3i/8}}.
$$

Since $d \geq 144$, we have that $d \geq 128 = 2^7$,

$$
\sum_{i=0}^{k} \binom{k}{i} \frac{2^{2i}}{d^{i/2}} \leq \sum_{i=0}^{k} \frac{e^i}{i^i} \frac{2^{2i}}{d^{3i/8}} \leq \sum_{i=0}^{k} \frac{e^i}{i^i} \frac{2^{2i}}{2^{21i/8}} = \sum_{i=0}^{k} \frac{e^i}{i^i} \frac{1}{2^{5i/8}}.
$$

Note that for $i \geq 4$, one has that $e^i/i^i \leq 1$. Hence,

$$
\sum_{i=0}^{k} \binom{k}{i} \frac{2^{2i}}{d^{i/2}} \leq \sum_{i=0}^{k} \frac{e^i}{i^i} \frac{1}{2^{5i/8}} \leq \sum_{i=0}^{3} \frac{e^i}{i^i} \frac{1}{2^{5i/8}} + \sum_{i=4}^{k} \frac{1}{2^{5i/8}}.
$$

We have that

$$\sum_{i=0}^{k} \frac{1}{2^{5i/8}} = \frac{2}{2-2^{3/8}}.$$

Hence,

$$\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{2^{2i}}{d^{i/2}} \leq \sum_{i=0}^{k} \binom{k}{i} \frac{2^{2i}}{d^{i/2}} \leq \sum_{i=0}^{3} \frac{e^i}{i^i} \frac{1}{2^{5i/8}} + \sum_{i=4}^{k} \frac{1}{2^{5i/8}} \leq \gamma_1.$$

Thus,

$$-\gamma_1 \leq -\sum_{i=0}^{k} \binom{k}{i} \frac{2^{2i}}{d^{i/2}} \leq -\sum_{i\in\mathrm{Odd}} \binom{k}{i} \frac{2^{2i}}{d^{i/2}},$$

and

$$-\gamma_1 \leq -\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{2^{2i}}{d^{i/2}} \leq -\sum_{i\in\mathrm{Even}\setminus\{\mathrm{k}\}} \binom{k-1}{i} \frac{2^{2i}}{d^{i/2}}.$$

Going back to the expanded expression (2.25),

$$\begin{aligned}
&2\left(1-\frac{4}{\sqrt{d}}\right)^k\left(\frac{k}{\sqrt{d}}-\frac{2k}{d}-\frac{80\sqrt{2}k^2}{d^{3/4}}\right)\\
&\frac{2k}{\sqrt{d}}-\frac{8k}{d}\sum_{i\in\mathrm{Even}\setminus\{k\}}\binom{k-1}{i}\frac{4^i}{d^{i/2}}-\frac{4k}{d}\sum_{i\in\mathrm{Even}}\binom{k}{i}\frac{4^i}{d^{i/2}}-\frac{160\sqrt{2}k^2}{d^{3/4}}\sum_{i\in\mathrm{Even}}\binom{k}{i}\frac{4^i}{d^{i/2}}\\
&\geq \frac{2k}{\sqrt{d}}-\frac{8k}{d}\gamma_1-\frac{4k}{d}\gamma_1-\frac{160\sqrt{2}k^2}{d^{3/4}}\gamma_1 = \frac{2k}{\sqrt{d}}-\gamma_1\left(\frac{12k}{d}+\frac{160\sqrt{2}k^2}{d^{3/4}}\right).
\end{aligned}$$

Note that $k^2/d^{3/4} = 1/d^{4/8} \geq 1/d^{7/8} = k/d$. Hence,

$$\begin{aligned}
&2\left(1-\frac{4}{\sqrt{d}}\right)^k\left(\frac{k}{\sqrt{d}}-\frac{2k}{d}-\frac{80\sqrt{2}k^2}{d^{3/4}}\right)\\
&\geq \frac{2k}{\sqrt{d}}-\gamma_1\left(\frac{12k}{d}+\frac{160\sqrt{2}k^2}{d^{3/4}}\right) \geq \frac{2k}{\sqrt{d}}-\gamma_1\left(\frac{12k^2}{d^{3/4}}+\frac{160\sqrt{2}k^2}{d^{3/4}}\right)\\
&= \frac{2k}{\sqrt{d}}-\frac{k^2}{d^{3/4}}\gamma_1(12+160\sqrt{2}) = \frac{2k}{\sqrt{d}}-\frac{k^2}{d^{3/4}}\gamma.
\end{aligned}$$

□

Finally, we can prove the main result of this chapter.

Theorem 43. (see [Srivastava and Trevisan, 2018, Theorem 1.1]) Let $\varepsilon > 0$. Let $d \geq 144 \geq 16 + \varepsilon$. Set $C_\varepsilon := \sqrt{16+\varepsilon}/(\sqrt{16+\varepsilon}-4)$ and set $C_d := 1 + 4/\sqrt{d}$. There is $\gamma \in \mathbb{R}_{++}$ satisfying the following. If $G = (V, E, w)$ is a connected weighted graph with n vertices and $dn/2$ edges such that $\mathrm{girth}(G) > 2d^{1/8} + 1$. Then

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - \gamma\frac{1}{d^{5/8}} - \max\{C_\varepsilon, C_d, 4\}\frac{1}{\sqrt{n}}.$$

Proof. Set $\gamma := (102\gamma_1 + 254)$, where γ_1 is the constant of Lemma 42. If (2.16a) does not hold, one can use the multiple $\left(\Delta_w(G)\right)^{-1}G$ of G since the ratio between the largest eigenvalue of the laplacian matrix and the second smallest eigenvalue of the laplacian matrix is the same for G and its multiple, hence we can assume (2.16a).

If (2.16b) does not hold, then by Corollary 32,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - \frac{4\sqrt{d}}{n}.$$

Since d is the average combinatorial degree, one has that $d \leq n-1 \leq n$ since each vertex has at most $n-1$ neighbors. Hence,

$$\begin{aligned}\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} &\geq 1 + \frac{4}{\sqrt{d}} - \frac{4\sqrt{d}}{n} \geq 1 + \frac{4}{\sqrt{d}} - \frac{4\sqrt{n}}{n} = 1 + \frac{4}{\sqrt{d}} - \frac{4}{\sqrt{n}} \\ &\geq 1 + \frac{4}{\sqrt{d}} - \max\{C_\varepsilon, C_d, 4\}\frac{1}{\sqrt{n}} - \gamma\frac{1}{d^{5/8}}.\end{aligned}$$

Therefore, we can assume (2.16b).

If (2.16c) does not hold, then by Lemma 33,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq 1 + \frac{4}{\sqrt{d}} - C_\varepsilon\frac{1}{n} \geq 1 + \frac{4}{\sqrt{d}} - C_\varepsilon\frac{1}{\sqrt{n}} \geq 1 + \frac{4}{\sqrt{d}} - \max\{C_\varepsilon, C_d, 4\}\frac{1}{\sqrt{n}} - \gamma\frac{1}{d^{5/8}}.$$

Hence, we can assume (2.16c).

If (2.16d) does not hold, then by Lemma 34,

$$\begin{aligned}\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} &\geq 1 + \frac{1}{\sqrt{4}} - \max\{C_\varepsilon, C_d\}\frac{1}{n} \geq 1 + \frac{1}{\sqrt{4}} - \max\{C_\varepsilon, C_d\}\frac{1}{\sqrt{n}} \\ &\geq 1 + \frac{4}{\sqrt{d}} - \max\{C_\varepsilon, C_d, 4\}\frac{1}{\sqrt{n}} - \gamma\frac{1}{d^{5/8}}.\end{aligned}$$

Hence, we can assume (2.16d). Therefore, we may assume the hypoteses in (2.16).

Set $k := d^{1/8}$ and note that $k < (\mathrm{girth}(G) - 1)/2$. For each $r \in V$ define $f_r : V \to \mathbb{R}$ as in (2.17). Let $r \in V$ be a vertex that satisfies Lemma 42. Define $f_r' : V \to \mathbb{R}$ as

$$f_r'(v) = \begin{cases} f_r(v) & \text{if } \mathrm{dist}(r, v) \text{ is even,} \\ -f_r(v) & \text{otherwise.} \end{cases} \tag{2.26}$$

Note that for each $uv \in V$ such that $\mathrm{dist}(r, u) \leq k$ and $\mathrm{dist}(r, v) \leq k$, we have that either $f_r(u) > 0$ and $f_r(v) < 0$, or $f_r(u) < 0$ and $f_r(v) > 0$, because, by Lemma 35, the subgraph induced by $B_k(r)$ is a tree. Hence, one has that $f_r'^T A_G f_r' = -f_r^T A_G f_r$. Recall that γ_1 satisfies:

$$f_r^T A_G f_r \geq \frac{2k}{\sqrt{d}} - \gamma_1\frac{k^2}{d^{3/4}},$$

and

$$f_r'^T A_G f_r' = -f_r^T A_G f_r \leq -\Big(\frac{2k}{\sqrt{d}} - \gamma_1\frac{k^2}{d^{3/4}}\Big).$$

Note that ${f_r'}^T D_G f_r' = \sum_{v\in V} D_G(v,v) f_r(v)^2 = f_r^T D_G f_r$. So,

$$f_r^T L_G f_r = f_r^T (D_G - A_G) f_r \le f_r^T D_G f_r - \left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right),$$

and

$${f_r'}^T L_G f_r' = {f_r'}^T (D_G - A_G) f_r' \ge {f_r'}^T D_G f_r' + \frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}} = f_r^T D_G f_r + \frac{2k}{\sqrt{d}} - M_1 \frac{k^2}{d^{3/4}}.$$

By (2.16a), we have that $D_G(v,v) \le 1$ for each $v \in V$. Hence, using Lemma 38,

$$f_r^T D_G f_r = \sum_{v\in V} D_G(v,v) f_r(v)^2 \le \sum_{v\in V} f_r(v)^2 = \|f_r\|^2 \le k+1.$$

Now we can bound the ratio

$$\begin{aligned}
\frac{{f_r'}^T L_G f_r'}{f_r^T L_G f_r} &\ge \frac{f_r^T D_G f_r + \frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}}{f_r^T D_G f_r - \left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right)} = \frac{f_r^T D_G f_r - \left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right) + 2\left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right)}{f_r^T D_G f_r - \left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right)} \\
&= 1 + \frac{2\left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right)}{f_r^T D_G f_r - \left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right)} \ge 1 + \frac{2\left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right)}{f_r^T D_G f_r} \\
&\ge 1 + \frac{2\left(\frac{2k}{\sqrt{d}} - \gamma_1 \frac{k^2}{d^{3/4}}\right)}{k+1} = 1 + \frac{2k \frac{2}{\sqrt{d}}\left(1 - \gamma_1 \frac{k}{2d^{1/4}}\right)}{k+1} \\
&= 1 + \frac{4k\left(1 - \gamma_1 \frac{k}{2d^{1/4}}\right)}{\sqrt{d}(k+1)} = 1 + \frac{4}{\sqrt{d}} \frac{k}{k+1}\left(1 - \gamma_1 \frac{k}{2d^{1/4}}\right) \\
&\ge 1 + \frac{4}{\sqrt{d}} \frac{k-1}{k}\left(1 - \gamma_1 \frac{k}{2d^{1/4}}\right) = 1 + \frac{4}{\sqrt{d}}\left(1 - \frac{1}{k}\right)\left(1 - \gamma_1 \frac{k}{2d^{1/4}}\right).
\end{aligned} \tag{2.27}$$

By Lemma 29,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \ge \frac{\frac{{f_r'}^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r'}{\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r'\|^2}}{\frac{f_r^T \mathrm{Proj}_{\{\mathbb{1}\}^\perp} L_G \mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r}{\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r\|^2}} = \frac{{f_r'}^T L_G f_r'}{f_r^T L_G f_r} \frac{\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r\|^2}{\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r'\|^2}$$

Note that $\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r'\|^2 \le \|f_r'\|^2 = \|f_r\|^2$. So, using Lemma 39,

$$\frac{\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r\|^2}{\|\mathrm{Proj}_{\{\mathbb{1}\}^\perp} f_r'\|^2} \ge \frac{\|f_r\|^2\left(1 - \frac{50}{d^2}\right)}{\|f_r\|^2} = \left(1 - \frac{50}{d^2}\right). \tag{2.28}$$

Combining (2.27), (2.28) and $k = d^{1/8}$,

$$\begin{aligned}\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} &\geq \left(1+\frac{4}{\sqrt{d}}\left(1-\frac{1}{k}\right)\left(1-\gamma_1\frac{k}{2d^{1/4}}\right)\right)\left(1-\frac{50}{d^2}\right). \\ &= \left(1+\frac{4}{\sqrt{d}}\left(1-\frac{1}{d^{1/8}}\right)\left(1-\gamma_1\frac{1}{2d^{1/8}}\right)\right)\left(1-\frac{50}{d^2}\right).\end{aligned}$$

One can show by expanding the expression that,

$$\frac{\lambda_n^{\uparrow}(L_G)}{\lambda_2^{\uparrow}(L_G)} \geq \left(1+\frac{4}{\sqrt{d}}\left(1-\frac{1}{d^{1/8}}\right)\left(1-\gamma_1\frac{1}{2d^{1/8}}\right)\right)\left(1-\frac{50}{d^2}\right) \geq 1+\frac{4}{\sqrt{d}}-\gamma\frac{1}{d^{5/8}}.$$

$\square$

Chapter 3

Irregular Expanders

3.1 Introduction

Another generalization of expander graphs uses the normalized Laplacian matrix. The result proved in this section uses the notion of *Weak Ramanujan graphs*, which satisfy some expander properties.

Definition 44. Let $G = (V, E)$ be a graph. Set

$$\sigma_G := 2\frac{\sum_{v \in V} \deg_G(v)\sqrt{\deg_G(v) - 1}}{\sum_{v \in V} \deg_G(v)^2}.$$

We call G a *weak Ramanujan graph* if

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \geq 1 - \sigma_G \geq \frac{1}{2}.$$

Note that for weak Ramanujan graphs that are d-regular graphs, one has

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \geq 1 - \sigma_G = 1 - 2\frac{\sum_{v \in V} d\sqrt{d-1}}{\sum_{v \in V} d^2} = 1 - 2\frac{\sqrt{d-1}}{d}.$$

Hence,

$$\lambda_2^{\downarrow}(A_G) = d - d\lambda_2^{\uparrow}(\mathcal{L}) \leq 2\sqrt{d-1},$$

which is one of the conditions on the definition of Ramanujan graphs (Definition 21).

In this chapter the graphs considered, unlike in chapter 2, are unweighted but can be irregular, as in chapter 2.

3.2 Preliminaries

Lemma 45. Let $G = (V, E)$ be a connected graph. Then

$$\lambda_2^{\uparrow}(\mathcal{L}_G) = \min_{f \perp D_G \mathbb{1}} \frac{f^T L_G f}{f^T D_G f}.$$

Proof. By Lemma 13 and Theorem 4,

$$\lambda_2^{\uparrow}(\mathcal{L}_G) = \min_{g \perp D^{1/2} \mathbb{1}} \frac{g^T \mathcal{L}_G g}{g^T g}.$$

Let $\psi \in \mathbb{R}^V$ such that $\psi \perp D_G^{1/2} \mathbb{1}$. Set $\varphi := D_G^{-1/2} \psi$. Note that $D_G^{1/2} = D_G D_G^{-1/2}$. So,

$$0 = \mathbb{1}^T D_G^{1/2} \psi = \mathbb{1}^T D_G D_G^{-1/2} \psi = \mathbb{1}^T D_G f = \varphi^T D_G \mathbb{1}.$$

Hence, one has that $\varphi \perp D_G \mathbb{1}$. So,

$$\begin{aligned}
\lambda_2^{\uparrow}(\mathcal{L}_G) &= \min_{g \perp D^{1/2} \mathbb{1}} \frac{g^T \mathcal{L}_G g}{g^T g} \\
&= \min_{g \perp D_G^{1/2} \mathbb{1}} \frac{g^T D_G^{-1/2} L_G D_G^{-1/2} g}{g^T g} \\
&= \min_{f \perp D_G \mathbb{1}} \frac{f^T L_G f}{f^T D_G^{1/2} D_G^{1/2} f} \\
&= \min_{f \perp D_G \mathbb{1}} \frac{f^T L_G f}{f^T D_G f}.
\end{aligned}$$

□

Lemma 46. If $G = (V, E)$ is a graph such that $G \neq K_V$, then

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \leq 1.$$

Proof. Let $G = (V, E)$ be a graph such that $G \neq K_V$. Then there exist $u, v \in V$ such that $u \notin N(v)$. Hence, define the vector $f := \deg_G(v) e_u - \deg_G(u) e_v$. Note that

$$\deg_G^T f = \deg_G(u) \deg_G(v) - \deg_G(v) \deg_G(u) = 0.$$

By Lemma 45,

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \leq \frac{f^T L_G f}{f^T D_G f} = \frac{\deg_G(v) \deg_G(u)^2 + \deg_G(u) \deg_G(v)^2}{\deg_G(v) \deg_G(u)^2 + \deg_G(v) \deg_G(u)^2} = 1.$$

□

Lemma 47. Let $G = (V, E)$ be a graph. Let $S \subseteq V$. Then

$$\mathbb{1}^T D_G \mathbb{1}_S = \mathrm{vol}(S).$$

Proof. We have that

$$\mathbb{1}^T D_G = \sum_{v \in V} e_v^T \deg_G(v).$$

So,

$$\mathbb{1}^T D_G \mathbb{1}_S = \sum_{v \in V} e_v^T \deg_G(v) \mathbb{1}_S = \sum_{v \in S} e_v^T \deg_G(v) = \mathrm{vol}(S). \qquad \square$$

Lemma 48. ([Chung Graham, 2016, Lemma 1]) Let $G = (V, E)$ be a graph such that $G \neq K_V$. Let $S \subseteq V$. Then

$$\frac{|\delta_G(S)|}{\mathrm{vol}(S)} \geq \lambda_2^{\uparrow}(\mathcal{L}_G)\Big(1 - \frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}\Big).$$

Proof. Set $\overline{S} := V \setminus S$ and

$$f := \frac{\mathbb{1}_S}{\mathrm{vol}(S)} - \frac{\mathbb{1}_{\overline{S}}}{\mathrm{vol}(\overline{S})} \in \mathbb{R}^V.$$

By Lemma 47,

$$\mathbb{1}^T D_G f = \frac{\mathbb{1}^T D_G \mathbb{1}_S}{\mathrm{vol}(S)} - \frac{\mathbb{1}^T D_G \mathbb{1}_{\overline{S}}}{\mathrm{vol}(\overline{S})} = \frac{\mathrm{vol}(S)}{\mathrm{vol}(S)} - \frac{\mathrm{vol}(\overline{S})}{\mathrm{vol}(\overline{S})} = 1 - 1 = 0.$$

So, we conclude that $f \perp D_G \mathbb{1}$. By Lemma 45,

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \leq \frac{f^T L_G f}{f^T D_G f}.$$

Note that if $v \in S$ then $f_v = 1/\mathrm{vol}(S)$, otherwise $f_v = 1/\mathrm{vol}(\overline{S})$. Hence,

$$f^T L_G f = \sum_{uv \in E} (f_u - f_v)^2 = \sum_{e \in \delta(S)} \Big(\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(\overline{S})}\Big)2 = |\delta(S)|\Big(\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(\overline{S})}\Big)^2,$$

and

$$f^T D_G f = \sum_{v \in V} d_v f_v^2 = \sum_{v \in S} d_v f_v^2 + \sum_{v \in \overline{S}} d_v f_v^2 = \frac{\sum_{v \in S} d_v}{\mathrm{vol}(S)^2} + \frac{\sum_{v \in \overline{S}} d_v}{\mathrm{vol}(\overline{S})^2} = \frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(\overline{S})}.$$

So,

$$\frac{f^T L_G f}{f^T D_G f} = \frac{\delta(S)\Big(\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(\overline{S})}\Big)^2}{\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(\overline{S})}} = \delta(S)\Big(\frac{1}{\mathrm{vol}(S)} + \frac{1}{\mathrm{vol}(\overline{S})}\Big) = \delta(S)\frac{\mathrm{vol}(S) + \mathrm{vol}(\overline{S})}{\mathrm{vol}(S)\mathrm{vol}(\overline{S})}.$$

Note that $\mathrm{vol}(S) + \mathrm{vol}(\overline{S}) = \mathrm{vol}(V)$. So,

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \leq \frac{f^T L_G f}{f^T D_G f} = \delta(S)\frac{\mathrm{vol}(S) + \mathrm{vol}(\overline{S})}{\mathrm{vol}(S)\mathrm{vol}(\overline{S})} = \delta(S)\frac{\mathrm{vol}(V)}{\mathrm{vol}(S)\mathrm{vol}(\overline{S})}.$$

Rearranging the terms,

$$\frac{\delta(S)}{\mathrm{vol}(S)} \geq \lambda_2^{\uparrow}(\mathcal{L}_G)\frac{\mathrm{vol}(\overline{S})}{\mathrm{vol}(V)},$$

but $\mathrm{vol}(\overline{S}) = \mathrm{vol}(V) - \mathrm{vol}(S)$. So,

$$\frac{\delta(S)}{\mathrm{vol}(S)} \geq \lambda_2^{\uparrow}(\mathcal{L}_G)\frac{\mathrm{vol}(\overline{S})}{\mathrm{vol}(V)} = \lambda_2^{\uparrow}(\mathcal{L}_G)\frac{\mathrm{vol}(V) - \mathrm{vol}(S)}{\mathrm{vol}(V)} = \lambda_2^{\uparrow}(\mathcal{L}_G)\Big(1 - \frac{\mathrm{vol}(S)}{\mathrm{vol}(V)}\Big). \qquad \square$$

Lemma 49. Let $G = (V, E)$ be a graph such that $G \neq K_V$. Let $S \subseteq V$. Then $\delta(S \cup N(S))$ and $\delta(S)$ are disjoint, and $\delta(N(S)) = \delta(S \cup N(S)) \cup \delta(S)$.

Proof. Suppose that there is $e \in E$ such that $e \in \delta(S \cup N(S))$ and $e \in \delta(S)$. Since $uv \in \delta(S)$, one of its end is in S and the other one is in $\delta(S)$. Hence both ends of e are in $S \cup N(S)$, a contradiction since $e \in \delta(S \cup N(S))$.

Let $uv \in \delta(N(S))$. We may assume that $u \in N(S)$, because either $u \in N(S)$ or $v \in N(S)$ and if $v \in N(S)$ we can relabel the vertices. So either $v \in V \setminus (S \cup N(S))$ or $v \in S$. If $v \in V \setminus (S \cup N(S))$, then $uv \in \delta(S \cup N(S)$, otherwise $v \in S$. So, one has that $uv \in \delta(S)$. Hence, we conclude that $\delta(N(S)) \subseteq \delta(S \cup N(S)) \cup \delta(S)$.

Let $uv \in \delta(S \cup N(S))$. We may assume that $u \in \delta(S \cup N(S))$. Note that $u \in N(S)$, otherwise $v \in N(S)$ because we would have $u \in S$, which would lead to $uv \notin \delta(S \cup N(S))$. Hence $u \in N(S)$ and $uv \in \delta(N(S))$.

Let $uv \in \delta(S)$. We may assume that $u \in \delta(S)$. Hence, one has that $v \in N(S)$, where it follows that $uv \in N(S)$. So we conclude that $\delta(S \cup N(S)) \cup \delta(S) \subseteq \delta(N(S))$. $\square$

Lemma 50. ([Chung Graham, 2016, Lemma 4]) Let $G = (V, E)$ be a connected graph such that $G \neq K_V$. Let $\varepsilon \in \mathbb{R}_{++}$ such that $\varepsilon \leq 1/2$. Let $S \subseteq V$ such that $\mathrm{vol}_G(S \cup N(S)) \leq \varepsilon \mathrm{vol}_G(V)$. Then

$$\frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)} \geq \frac{2\lambda_2^{\uparrow}(\mathcal{L}_G)}{1 - \lambda_2^{\uparrow}(\mathcal{L}_G) + 2\varepsilon}, \tag{3.1}$$

and, if $1/2 \leq \lambda_2^{\uparrow}(\mathcal{L}_G) \leq 1 - 2\varepsilon$, then

$$\frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)} \geq \frac{1}{(1 - \lambda_2^{\uparrow}(\mathcal{L}_G) + 2\varepsilon)^2}. \tag{3.2}$$

Proof. We first prove (3.1). Note that $\mathrm{vol}_G(N(S)) \geq |\delta(N(S))|$, because each edge in $N(S)$ has one of its end in S. So, each edge is counted once in $\mathrm{vol}_G(N(S))$. By Lemma 49, we have that $\delta(N(S)) = \delta(S \cup N(S)) \cup \delta(S)$, and that $\delta(S \cup N(S))$ and $\delta(S)$ are disjoint. So $|\delta(S \cup N(S)) \cup \delta(S)| = |\delta(S \cup N(S))| + |\delta(S)|$. Hence,

$$\mathrm{vol}_G(N(S)) \geq |\delta(N(S))| = |\delta(S \cup N(S))| + |\delta(S)|. \tag{3.3}$$

Applying Lemma 48 with the set $S \cup N(S)$,

$$\frac{|\delta(S \cup N(S))|}{\mathrm{vol}_G(S \cup N(S))} \geq \lambda_2^{\uparrow}(\mathcal{L}_G)\Big(1 - \frac{\mathrm{vol}_G(S \cup N(S))}{\mathrm{vol}_G(V)}\Big).$$

By hypothesis, we have that $\varepsilon \geq \mathrm{vol}_G(S \cup N(S))/\mathrm{vol}_G(V)$. So,

$$\frac{|\delta(S \cup N(S))|}{\mathrm{vol}_G(S \cup N(S))} \geq \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)\Big(1 - \frac{\mathrm{vol}_G(S \cup N(S))}{\mathrm{vol}_G(V)}\Big) \geq \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon).$$

Hence, we have that $|\delta(S \cup N(S))| \geq \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\,\mathrm{vol}_G(S \cup N(S))$. Since S and $N(S)$ are disjoint, we have that $\mathrm{vol}_G(S \cup N(S)) = \mathrm{vol}_G(S) + \mathrm{vol}_G(N(S))$. So,

$$|\delta(S \cup N(S))| \geq \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)(\mathrm{vol}_G(S) + \mathrm{vol}_G(N(S)). \tag{3.4}$$

Applying Lemma 48 with the set S,

$$\frac{|\delta(S)|}{\mathrm{vol}_G(S)} \geq \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)\Big(1 - \frac{\mathrm{vol}_G(S)}{\mathrm{vol}_G(V)}\Big).$$

Since $S \subseteq S \cup N(S)$, we have that $\mathrm{vol}_G(S) \leq \mathrm{vol}_G(S \cup N(S))$. Hence, we have that $\varepsilon \geq \mathrm{vol}_G(S)/\mathrm{vol}_G(V)$. So,

$$|\delta(S)| \geq \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\,\mathrm{vol}_G(S). \tag{3.5}$$

Combining (3.3), (3.4) and (3.5),

$$\begin{aligned}\mathrm{vol}_G(N(S)) &\geq |\delta(S \cup N(S))| + |\delta(S)| \\ &\geq \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)(\mathrm{vol}_G(S) + \mathrm{vol}_G(N(S)) + \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\,\mathrm{vol}_G(S) \\ &= 2\lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\,\mathrm{vol}_G(S) + \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\,\mathrm{vol}_G(N(S)).\end{aligned}$$

Rearranging the terms,

$$\mathrm{vol}_G(N(S))\big(1 - \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\big) \geq 2\lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\,\mathrm{vol}_G(S).$$

Since $\lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G) \leq 1$ and $\varepsilon > 0$, we have that $1 - \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon) > 0$. So,

$$\mathrm{vol}_G(N(S)) \geq \frac{2\lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)\,\mathrm{vol}_G(S)}{1 - \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)}.$$

Hence,

$$\frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)} \geq \frac{2\lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)}{1 - \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)(1-\varepsilon)} \geq \frac{2\lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G)}{1 - \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G) + 2\varepsilon},$$

where the last inequality holds if $\varepsilon \in (0, 1/2]$. Thus, (3.1) is proved.

Set $\gamma := 1 - \lambda_2^{\uparrow}(\boldsymbol{\mathcal{L}}_G) \geq 0$, set $f := \mathbb{1}_S + \gamma\mathbb{1}_{N(S)} \in \mathbb{R}_+^V$, set $c := \deg_G^T f/\mathrm{vol}_G(V) \geq 0$ and set $g := f - c\mathbb{1} \in \mathbb{R}^V$. We have that

$$c^2 = \frac{\big(\deg_G^T f\big)^2}{\mathrm{vol}_G(G)^2} = \frac{\Big(\sum_{v \in V} f(v)\deg_G(v)\Big)^2}{\mathrm{vol}_G(V)^2} = \frac{\Big(\sum_{v \in V} f(v)\sqrt{\deg_G(v)}\sqrt{\deg_G(v)}\Big)^2}{\mathrm{vol}_G(V)^2}.$$

Using Cauchy-Schwartz inequality,

$$\begin{aligned} c^2 &= \frac{\left(\sum_{v\in V} f(v)\sqrt{\deg_G(v)}\sqrt{\deg_G(v)}\right)^2}{\mathrm{vol}_G(V)^2} \\ &\le \frac{1}{\mathrm{vol}_G(V)^2}\sum_{v\in V} f(v)^2\sqrt{\deg_G(v)}^2 \sum_{v\in V}\sqrt{\deg_G(v)}^2 \\ &= \frac{1}{\mathrm{vol}_G(V)^2}\sum_{v\in V} f(v)^2\deg_G(v)\sum_{v\in V}\deg_G(v). \end{aligned}$$

Since $f(v) \neq 0$ if $v \in S \cup N(S)$,

$$c^2 \le \frac{1}{\mathrm{vol}_G(V)^2}\sum_{v\in V} f(v)^2\deg_G(v)\sum_{v\in V}\deg_G(v) = \frac{\mathrm{vol}_G(S\cup N(S))}{\mathrm{vol}_G(V)^2}\sum_{v\in V} f(v)^2\deg_G(v).$$

By hypothesis, one has that $\varepsilon \ge \mathrm{vol}_G(S\cup N(S))/\mathrm{vol}_G(V)$. So,

$$\begin{aligned} c^2 &\le \frac{\mathrm{vol}_G(S\cup N(S))}{\mathrm{vol}_G(V)^2}\sum_{v\in V} f(v)^2\deg_G(v) \\ &\le \frac{\varepsilon}{\mathrm{vol}_G(V)}\sum_{v\in V} f(v)^2\deg_G(v) \qquad (3.6) \\ &= \frac{\varepsilon}{\mathrm{vol}_G(V)} f^T D_G f. \end{aligned}$$

Note that $\left(\mathbb{1}^T A_G\right)_v = \sum_{u\in V} A_G(u,v) = \sum_{u\in V}[uv \in E] = \deg(v)$ for each $v \in V$. Whence $\mathbb{1}^T A_G = \deg_G^T$. Also, since $c = \deg_G^T f/\mathrm{vol}_G(V)$, we have that $\deg_G^T f = c\,\mathrm{vol}_G(V)$. So,

$$\begin{aligned} g^T A_G g &= (f - c\mathbb{1})^T A_G (f - c\mathbb{1}) = f^T A_G f - 2c\mathbb{1}^T A_G f + c^2\mathbb{1}^T A_G\mathbb{1} \\ &= f^T A_G f - 2c\deg_G^T f + c^2\deg_G^T\mathbb{1} = f^T A_G f - 2c^2\,\mathrm{vol}_G(V) + c^2\,\mathrm{vol}_G(V) \\ &= f^T A_G - c^2\,\mathrm{vol}_G(V). \end{aligned}$$

Hence, we have that $f^T A_G f = g^T A_G g + c^2\,\mathrm{vol}_G(V)$. Note that

$$g^T D_G\mathbb{1} = f^T\deg_G - c\mathbb{1}^T\deg_G = f^T\deg_G - \mathrm{vol}_G(V)\deg_G^T f/\mathrm{vol}_G(V) = f^T\deg_G - f^T\deg_G = 0.$$

By Lemma 45,

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \le g^T L_G g/g^T D_G g = (g^T D_G g - g^T A_G g)/g^T D_G g.$$

Rearranging the terms, we have that $g^T A_G g \le (1 - \lambda_2^{\uparrow}(\mathcal{L}_G))g^T D_G g = \gamma g^T D_G g$. Hence,

$$f^T A_G f = g^T A_G g + c^2\,\mathrm{vol}_G(V) \le \gamma g^T D_G g + c^2\,\mathrm{vol}_G(V).$$

Note that

$$\begin{aligned} g^T D_G g &= (f - c\mathbb{1})^T D_G (f - c\mathbb{1}) = f^T D_G f - 2cf^T D_G \mathbb{1} + c^2 \mathbb{1}^T D_G \mathbb{1} \\ &= f^T D_G f - 2cf^T \deg_G + c^2 \operatorname{vol}_G(V) = f^T D_G f - 2c^2 \operatorname{vol}_G(V) + c^2 \operatorname{vol}_G(V) \\ &= f^T D_G f - c^2 \operatorname{vol}_G(V). \end{aligned}$$

So,

$$\begin{aligned} f^T A_G f &\leq \gamma g^T D_G g + c^2 \operatorname{vol}_G(V) \\ &= \gamma\big(f^T D_G f - c^2 \operatorname{vol}_G(V)\big) + c^2 \operatorname{vol}_G(V) \\ &= \gamma f^T D_G f + (1-\gamma)c^2 \operatorname{vol}_G(V). \end{aligned}$$

Using (3.6),

$$f^T A_G f \leq \gamma f^T D_G f + (1-\gamma)c^2 \operatorname{vol}_G(V) \leq \gamma f^T D_G f + (1-\gamma)\varepsilon f^T D_G f.$$

Since $\gamma \geq 0$, we have that $1 - \gamma \leq 1$. So,

$$f^T A_G f \leq \gamma f^T D_G f + (1-\gamma)\varepsilon f^T D_G f \leq \gamma f^T D_G f + \varepsilon f^T D_G f = (\gamma + \varepsilon) f^T D_G f.$$

Note that

$$\begin{aligned} f^T D_G f &= \sum_{v \in V} \deg_G(v) f(v)^2 = \sum_{v \in S} \deg_G(v) f(v)^2 + \sum_{v \in N(S)} \deg_G(v) f(v)^2 \\ &= \sum_{v \in S} \deg_G(v) + \sum_{v \in N(S)} \deg_G(v)\gamma^2 = \operatorname{vol}_G(S) + \gamma^2 \operatorname{vol}_G(N(S)). \end{aligned}$$

Hence,

$$f^T A_G f \leq (\gamma + \varepsilon) f^T D_G f = (\gamma + \varepsilon)\big(\operatorname{vol}_G(S) + \gamma^2 \operatorname{vol}_G(N(S))\big). \tag{3.7}$$

Let $H := G[S]$. So,

$$\begin{aligned} f^T A_G f &= 2 \sum_{uv \in E} f(u)f(v) = 2\Big(\sum_{uv \in E[H]} f(u)f(v) + \sum_{uv \in \delta_G(S)} f(u)f(v)\Big) \\ &= 2\Big(\sum_{uv \in E[H]} + \sum_{uv \in \delta_G(S)} \gamma\Big) = 2(|E[H]| + \gamma|\delta_G(S)|). \end{aligned}$$

Since $\lambda_2^{\uparrow}(\mathcal{L}_G) \geq 1/2$, we have that $1 - 2\gamma = 1 - 2 + 2\lambda_2^{\uparrow}(\mathcal{L}_G) = 2\lambda_2^{\uparrow}(\mathcal{L}_G) - 1 \geq 0$. Note that $\operatorname{vol}_G(S) = |\delta_G| + 2|E[H]|$. So,

$$\begin{aligned} f^T A_G f &= 2\big(|E[H]| + \gamma|\delta_G(S)|\big) = 2\big(|E[H]| + \gamma \operatorname{vol}_G(S) - \gamma 2|E[H]|\big) \\ &= 2\big(|E[H]|(1-2\gamma) + \gamma \operatorname{vol}_G(S)\big) \geq 2\gamma \operatorname{vol}_G(S). \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8),

$$2\gamma \operatorname{vol}_G(S) \leq (\gamma + \varepsilon)\big(\operatorname{vol}_G(S) + \gamma^2 \operatorname{vol}_G(N(S))\big).$$

Rearranging the terms,

$$\begin{aligned}
&\frac{2\gamma}{\gamma+\varepsilon} \leq 1+\gamma^2\frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)}\\
\Rightarrow&\frac{2\gamma}{\gamma+\varepsilon}-1 \leq \gamma^2\frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)}\\
\Rightarrow&\frac{2\gamma-\gamma-\varepsilon}{\gamma+\varepsilon} \leq \gamma^2\frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)}\\
\Rightarrow&\frac{\gamma-\varepsilon}{\gamma^2(\gamma+\varepsilon)} \leq \frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)}.
\end{aligned}$$

Since $\lambda_2^{\uparrow}(\mathcal{L}_G) \leq 1-2\varepsilon$, we have that $\gamma = 1-\lambda_2^{\uparrow}(\mathcal{L}_G) \geq 1-1+2\varepsilon = 2\varepsilon$. We claim that $(\gamma-\varepsilon)/\gamma^2(\gamma+\varepsilon) \geq 1/(\gamma+2\varepsilon)^2$. So,

$$\begin{aligned}
&\frac{\gamma-\varepsilon}{\gamma^2(\gamma+\varepsilon)} \geq \frac{1}{(\gamma+2\varepsilon)^2}\\
\Rightarrow&(\gamma-\varepsilon)(\gamma+2\varepsilon)^2 \geq \gamma^2(\gamma+\varepsilon)\\
\Rightarrow&(\gamma-\varepsilon)(\gamma^2+4\gamma\varepsilon+4\varepsilon^2) \geq \gamma^3+\gamma^2\varepsilon\\
\Rightarrow&\gamma^3+4\gamma^2\varepsilon+4\gamma\varepsilon^2-\varepsilon\gamma^2-4\gamma\varepsilon^2-4\varepsilon^3 \geq \gamma^3+\gamma^2\varepsilon\\
\Rightarrow&4\gamma^2\varepsilon-\varepsilon\gamma^2-4\varepsilon^3 \geq \gamma^2\varepsilon\\
\Rightarrow&2\gamma^2\varepsilon \geq 4\varepsilon^3\\
\Rightarrow&\gamma^2 \geq 2\varepsilon^2.
\end{aligned}$$

Which is true because $\gamma \geq 2\varepsilon \Rightarrow \gamma^2 \geq 4\varepsilon^2 \geq 2\varepsilon^2$. So,

$$\frac{\mathrm{vol}_G(N(S))}{\mathrm{vol}_G(S)} \geq \frac{\gamma-\varepsilon}{\gamma^2(\gamma+\varepsilon)} \geq \frac{1}{(\gamma+2\varepsilon)^2} = \frac{1}{(1-\lambda_2^{\uparrow}(\mathcal{L}_G)+2\varepsilon)^2},$$

and (3.2) is proved. □

3.2.1 Weak Ramanujan Graphs

Theorem 51. ([Chung Graham, 2016, Theorem 6]) Let $\varepsilon > 0$. Set $c := 1/\ln(1.5)$. Let $G = (V, E)$ be a weak Ramanujan connected graph. Suppose that $\mathrm{vol}_G(V) \geq c\sigma_G^{\ln(\sigma_G)}/\varepsilon$. Then

$$\mathrm{diam}(G) \leq \left\lceil (1+\varepsilon)\frac{2\ln(\mathrm{vol}_G(V))}{\ln(\sigma_G^{-1})} \right\rceil$$

Proof. Set

$$t := \left\lceil (1+\varepsilon)\frac{\ln(\mathrm{vol}_G(V))}{\ln(\sigma_G^{-1})} \right\rceil.$$

Suppose that for every vertex $v \in V$, one has that $\mathrm{vol}_G(B_t(v)) > \mathrm{vol}_G(V)/2$. Let $u, v \in V$. Note that $\mathrm{vol}_G(B_t(v)) + \mathrm{vol}_G(B_t(u) > vol_G(V)$. Hence, there is a vertex $x \in \mathrm{vol}_G(B_t(v))$ and

$x \in \mathrm{vol}_G(B_t(u))$. Using the vertex x, we have a path from u to v with length at most

$$2\left\lceil (1+\varepsilon)\frac{\ln(\mathrm{vol}_G(V))}{\ln(\sigma_G^{-1})} \right\rceil.$$

So, it suffices to prove that for every vertex $v \in V$, one has that $\mathrm{vol}_G(B_t(v)) > \mathrm{vol}_G(V)/2$.

Suppose for the sake of contradiction that there exists $v \in V$ such that $\mathrm{vol}_G(B_t(v)) \le \mathrm{vol}_G(V)/2$. Set

$$s_j := \frac{\mathrm{vol}_G(B_j(v))}{\mathrm{vol}_G(V)}, \quad \text{for each } j \in \mathbb{N}.$$

In particular, we have that $s_t \le 1/2$, by assumption. Let $j \le t-1$ be a nonnegative integer. Since $B_j(v) \subseteq B_t(v)$,

$$s_{j+1} = \frac{\mathrm{vol}_G(B_{j+1}(v))}{\mathrm{vol}_G(V)} \le \frac{\mathrm{vol}_G(B_t(v))}{\mathrm{vol}_G(V)} = s_t \le \frac{1}{2}.$$

Recall that G is a weak Ramanujan graph. Hence, we have that $\lambda_2^{\uparrow}(\mathcal{L}_G) \ge 1/2$. Since $B_{j+1}(v) = B_j(v) \cup N(B_j(v))$, we can apply (3.1) of Lemma 50 with $\varepsilon = 1/2$ and $S = B_j(v)$. So,

$$\frac{\mathrm{vol}_G(N(B_j(v)))}{\mathrm{vol}_G(B_j(v))} \ge \frac{2\lambda_2^{\uparrow}(\mathcal{L}_G)}{1-\lambda_2^{\uparrow}(\mathcal{L}_G+2\varepsilon)} \ge \frac{2\frac{1}{2}}{1-\frac{1}{2}+2\frac{1}{2}} = \frac{2}{3} \ge 1/2.$$

Hence, one has that $\mathrm{vol}_G(N(B_j(v))) \ge 1/2\,\mathrm{vol}_G(B_j(v))$. Whence

$$\begin{aligned} s_{j+1} &= \frac{\mathrm{vol}_G(B_{j+1}(v))}{\mathrm{vol}_G(V)} = \frac{\mathrm{vol}_G(B_j(v) \cup N(B_j(v)))}{\mathrm{vol}_G(V)} = \frac{\mathrm{vol}_G(B_j(v)) + \mathrm{vol}_G(N(B_j(v)))}{\mathrm{vol}_G(V)} \\ &\ge \frac{\mathrm{vol}_G(B_j(v)) + 1/2\,\mathrm{vol}_G(B_j(v))}{\mathrm{vol}_G(V)} = 3/2\frac{\mathrm{vol}_G(B_j(v))}{\mathrm{vol}_G(v)} = 3/2 s_j. \end{aligned}$$

Set $c_1 := 4(\ln(3/2))^{-1}$. If $j \le t - c_1\ln(\sigma_G^{-1})$, then $1 \ge s_t \ge (3/2)^{c_1 \ln(\sigma_G^{-1})} s_j$. Note that

$$c_1 \ln(\sigma_G^{-1}) = 4(\ln(3/2))^{-1}\ln(\sigma_G^{-1}) = 4\frac{\ln(\sigma_G^{-1})}{\ln(3/2)} = 4\log_{3/2}(\sigma_G^{-1}) = \log_{3/2}(\sigma_G^{-4}).$$

Hence, one has that $1 \ge s_t \ge (3/2)^{c_1\ln(\sigma_G^{-1})} s_j = (3/2)^{\log_{3/2}(\sigma_G^{-4})} s_j = \sigma_G^{-4} s_j$, whence

$$\sigma_G^4 \ge s_j. \tag{3.9}$$

[Chung Graham, 2016, page 9] claims that (3.2) of Lemma 50 can be applied for $j \le t - c_1\ln(\sigma_G^{-1})$ (although it is not obvious if the conditions for applying this lemma were met). So, take $\varepsilon = s_{j+1}$ and $S = B_j(v)$,

$$\frac{s_{j+1}}{s_j} = \frac{\mathrm{vol}_G(B_{j+1}(v))}{\mathrm{vol}_G(B_j(v))} \ge \frac{\mathrm{vol}_G(N(B_j(v)))}{\mathrm{vol}_G(B_j(v))} \ge \frac{1}{(1-\lambda_2^{\uparrow}(\mathcal{L}_G)+2s_{j+1})^2}.$$

Since G is weak Ramanujan, we have that $1-\lambda_2^{\uparrow}(\mathcal{L}_G) \le \sigma_G$. By (3.9), we have that $s_{j+1} \le \sigma_G^4$

for each $j \leq t - c_1 \ln(\sigma_G^{-1}) - 1$. Hence,

$$\frac{s_{j+1}}{s_j} \geq \frac{1}{(1 - \lambda_2^{\uparrow}(\mathcal{L}_G) + 2s_{j+1})^2} \geq \frac{1}{(\sigma_G + \sigma_G^4)^2}.$$

Let $\ell \leq t - c_1 \ln(\sigma_G^{-1}) - 1$. We have that

$$\frac{s_\ell}{s_0} \geq \prod_{0 \leq j < \ell} \frac{1}{(\sigma_G + \sigma_G^4)^2} \geq \frac{1}{(\sigma_G + \sigma_G^4)^{2\ell}}.$$

[Chung Graham, 2016, page 9] claims that, since $s_0 = B_0(u)/\operatorname{vol}_G(V) = 1/\operatorname{vol}_G(V)$ and $s_\ell \leq s_t \leq 1/2$ one can prove that

$$\operatorname{vol}_G(V) \geq \frac{1}{\sigma_G^{2\ell}(1 + 2\sigma_G^4)^{2\ell}},$$

which implies that

$$\ell \leq \frac{\log(\operatorname{vol}_G(V))}{\ln(\sigma_G^{-1}) + 2\sigma_G^4}.$$

[Chung Graham, 2016, page 9] claims that, using the inequality

$$(1+\varepsilon)\frac{\log(\operatorname{vol}_G(V))}{\log(\sigma_G^{-1})} \leq t \leq c_1 \log(\sigma_G^{-1}) + \frac{\log(\operatorname{vol}_G(V))}{\log(\sigma_G^{-1}) + 2\sigma^4}$$

and with $\operatorname{vol}_G(V) \geq \sigma_G^{2c_1 \log(\sigma_G)}/\varepsilon$, ones has a contradiction. Hence $s_t \geq 1/2$. □

Theorem 52. ([Chung Graham, 2016, Theorem 7]) Let $\varepsilon \in [0, 1/2]$. Set $c := 4/\ln(1.5)$. Let $G = (V, E)$ be a connected weak Ramanujan graph. Let $l \leq \operatorname{diam}(G)/4$. If $\operatorname{diam}(G) \geq c \ln(\varepsilon^{-1})$. Then, for every $v \in V$,

$$B_l(v) \leq \varepsilon \operatorname{vol}_G(V).$$

Proof. Set $k := \operatorname{diam}(G)$. Suppose for the sake of contradiction that for $j_0 := \lceil k/4 \rceil$, there is a vertex $v \in V$ such that $\operatorname{vol}_G(B_v(j_0)) > \varepsilon \operatorname{vol}_G(V)$. Denote

$$s_j := \frac{\operatorname{vol}_G(B_v(r))}{\operatorname{vol}_G(V)} \quad \text{for each } j \in \mathbb{Z}.$$

Hence, we have that $s_{j_0} > \varepsilon$. Let r denote the least integer such that $s_r > 1/2$. [Chung Graham, 2016, page 9] claims that, by assumption, one has $r > k/4$.

Suppose that $r \geq k/2$. Note that $s_{r-1} \leq 1/2$. So, for each $j \leq r - 1$ one has that $s_j \leq 1/2$. So, we can apply (3.1) of Lemma 50 for each $j \leq k/2 - 2 \leq r - 2$ with $\varepsilon = 1/2$ and $S = B_v(j)$. Hence,

$$\frac{\operatorname{vol}_G(N(B_v(j)))}{\operatorname{vol}_G(B_v(j))} \geq \frac{2\lambda_2^{\uparrow}(\mathcal{L}_G)}{1 - \lambda_2^{\uparrow}(\mathcal{L}_G) + 2\varepsilon} \geq \frac{2}{3} \geq 1/2.$$

So $s_{j+1} \geq 1.5 s_j$ for each $j \leq k/2 - 2$. If $j \leq k/2 - c_1 \ln(\varepsilon^{-1})$ with $c_1 := 1/\ln(1.5)$, then

$$1 \geq s_{k/2} \geq (1.5)^{c_1 \ln(\varepsilon^{-1})} s_j = (1.5)^{\ln(\varepsilon^{-1})/\ln(1.5)} s_j = (1.5)^{\log_{1.5}(\varepsilon^{-1})} s_j = \varepsilon^{-1} s_j.$$

Whence $s_j \leq \varepsilon$. Since $k/4 \leq k/2 - c_1 \ln(\varepsilon^{-1})$, we have a contradiction because $s_{j_0} > \varepsilon$ and $s_{j_0} \leq s_{k/2 - c_1 \ln(\varepsilon^{-1})}$.

Suppose $r < k/2$. Define

$$\overline{s}_{j+1} := \frac{\mathrm{vol}_G(V \setminus B_j(v))}{\mathrm{vol}_G(V)} \quad \text{for each } j \in \mathbb{Z}_{++}.$$

Since $r < k/2$, for each $j \geq k/2$, we have that $s_j > 1/2$. So, one has $\overline{s}_{j+1} < 1/2$, for each $j \geq k/2$, because $\mathrm{vol}_G(V \setminus B_j(v)) = \mathrm{vol}_G(V) - \mathrm{vol}_G(B_j(v))$.

Suppose that $s_j \geq \varepsilon$ for each $j \leq k/2$. Using (3.1) of Lemma 50, for each $j \in \mathbb{Z}_{++}$ such that $r \leq j \leq k/2$, with $\varepsilon = 1/2$ and $S = V \setminus B_j(v)$,

$$\frac{\mathrm{vol}_G(N(V \setminus B_j(v)))}{\mathrm{vol}_G(V \setminus B_j(v))} \geq \frac{2\lambda_2^{\uparrow}(\mathcal{L}_G)}{1 - \lambda_2^{\uparrow}(\mathcal{L}_G) + 2\varepsilon} \geq 2/3 \geq 1/2.$$

Note that $N(V \setminus B_j(v)) = V \setminus B_{j-1}(v)$. Hence, one has $\overline{s}_j \geq 1.5\overline{s}_{j+1}$. Let $j_1 \geq k/2 - c_1 \ln(\varepsilon^{-1})$. Hence,

$$\overline{s}_{j_1+1} \geq \overline{s}_{k/2}(1.5)^{c_1 \ln(\varepsilon^{-1}/2)} = \overline{s}_{k/2}(1.5)^{\log_{1.5}(\varepsilon^{-1}/2)} = \overline{s}_{k/2}\varepsilon^{-1}.$$

By assumption, we have that $s_{k/2} \geq \varepsilon$. So, one has $\overline{s}_{j_1+1} \geq \varepsilon\varepsilon^{-1} = 1 \geq 1/2$. Hence, one has $s_{j_1} \leq 1/2$ for $j_1 \geq k/2 - c_1 \ln(\varepsilon^{-1})$. We can apply (3.1) of Lemma 50, for $j \geq j_1$, with $\varepsilon = 1/2$ and $S = B_j(v)$,

$$\frac{\mathrm{vol}_G(N(B_j(v)))}{\mathrm{vol}_G(B_j(v))} \geq \frac{2\lambda_2^{\uparrow}(\mathcal{L}_G)}{1 - \lambda_2^{\uparrow}(\mathcal{L}_G) + 2\varepsilon} \geq 2/3 \geq 1/2.$$

So, we conclude that $s_{j+1} \geq 1.5 s_j$. Let $j_2 \leq j_1 - c_1 \ln(\varepsilon^{-1})$. Hence,

$$1 \geq s_{j_1} \geq s_j(1.5)^{c_1 \ln(\varepsilon^{-1})} = s_j \varepsilon^{-1}.$$

Whence $s_j \leq \varepsilon$. Note that $j_1 - c_1 \ln(\varepsilon^{-1}) \geq k/2 - 2c_1 \ln(\varepsilon^{-1}) \geq k/4$, so we have a contradiction because $s_{j_0} > \varepsilon$ and $s_{j_0} \leq s_{j_1 - c_1 \ln(\varepsilon^{-1})}$.

Now, suppose that $\overline{s}_j < \varepsilon$ for $j \geq k/2$. [Chung Graham, 2016, page 10] claims that one can apply (3.2) of Lemma 50 for $j \geq k/2$. So, take $S = V \setminus B_j(v)$

$$\frac{\mathrm{vol}_G(N(V \setminus B_j(v)))}{\mathrm{vol}_G(V \setminus B_j(v))} \geq \frac{1}{(1 - \lambda_2^{\uparrow}(\mathcal{L}_G) + 2\varepsilon)^2} \geq \frac{1}{(\sigma + 2\varepsilon)^2}.$$

Hence,

$$\frac{\overline{s}_j}{\overline{s}_{j+1}} \geq \frac{1}{(\sigma + 2\varepsilon)^2}.$$

Let $j_3 = \lceil k/2 \rceil$. So,

$$\frac{\overline{s}_{j_3}}{\overline{s}_k} \geq \prod_{k/2 < j \leq k} \frac{1}{(\sigma_G + 2\varepsilon)^2} \geq \frac{1}{(\sigma_G + 2\varepsilon)^k}.$$

Note that $\overline{s}_k \geq 1/\operatorname{vol}_G(V)$. So,

$$\overline{s}_{j_3} \geq \frac{1}{\operatorname{vol}_G(V)(\sigma_G + 2\varepsilon)^k}.$$

[Chung Graham, 2016, page 10] claims that, using the inequality

$$\overline{s}_{j_3} \geq \frac{1}{\operatorname{vol}_G(V)(\sigma_G + 2\varepsilon)^k},$$

and the fact that $\overline{s}_{j_3} < \varepsilon$, one can prove that

$$k \geq \frac{\log(n) + \log(\varepsilon^{-1})}{\log(\sigma_G^{-1})}.$$

Also, [Chung Graham, 2016, page 10] claims that, using Lemma 50, for $j = k/2 - j' \geq r$ one has that

$$\overline{s}_j \geq \frac{1}{\operatorname{vol}_G(V)(\sigma_G + 2\varepsilon)^{k+2j'}},$$

which implies that, for some $j \leq k/2 - \log(\varepsilon^{-1})/\log(\sigma_G^{-1})$, one has $\overline{s}_j \geq 1/2$, and so $r \geq k/2 - \log(\varepsilon^{-1})/\log(\sigma^{-1})$.

Finally, [Chung Graham, 2016, page 10] claims that, for some $j \leq r - c_1\log(\varepsilon^{-1}) \leq k/2 - \log(\varepsilon^{-1})/\log(\sigma^{-1}) - c_1\log(\varepsilon^{-1})$, one has $s_j \leq \varepsilon$. Since $\log(\varepsilon^{-1})/\log(\sigma^{-1}) + c_1\log(\varepsilon^{-1}) < k/4$, we have a contradiction. □

3.2.2 Non-Backtracking Walks

Let $G = (V, E)$ be a graph. Recall that a non-backtracking walk is a sequence of vertices $p := \{v_i\}_{i=0}^t \subseteq V$ for some $t \in \mathbb{Z}_+$ such that $v_{i-1} \in N(v_i)$, for each $i \in [t-1]$, and $v_{i-1} \neq v_{i+1}$, for each $i \in [t-2]$. So if we consider a walk, at each step we want to go from a vertex to its neighbor without repeating the vertex at the previous step. Denote by $\mathcal{P}_{u,v}^{(k)}$ the set of non-backtracking walks from u to v with length k. Define the *modified transition probability matrix* $\tilde{P}_k$, for each $k \in \{0, \ldots, t-1\}$, as

$$\tilde{P}_k(u,v) := \begin{cases} I & \text{if } k = 0 \\ \sum_{p \in \mathcal{P}_{u,v}^{(k)}} \tilde{w}(p) & \text{if } k \geq 1, \end{cases} \quad \text{for each } u, v \in V, \tag{3.10}$$

where we define the weight function $\tilde{w}(p)$ of a non-backtracking walk $p := \{v_i\}_{i=0}^t \subseteq V$ with $t \geq 1$ as

$$\tilde{w}(p) := \frac{1}{\deg_G(v_0)\prod_{i=1}^{t-1}(\deg_G(v_i) - 1)}. \tag{3.11}$$

If the non-backtracking walk p is a sequence with a single vertex, i.e., its length is equal to 0, we define $\tilde{w}(p) := 1$. Define $\hat{E}$ as the set of directed edges obtained from the original edge set E, where for each undirected edge $uv \in E$, we create two directed arcs: (u, v) and (v, u). Formally, $\hat{E} := \bigcup_{uv \in E} \{(u, v), (v, u)\}$. Also, we define the transition probability matrix P for a random walk in $\hat{E}$ as

$$P((u, v), (u', v')) = [v = u'][u \neq v'] \frac{1}{\deg_G(v) - 1}.$$

The matrix P is indexed by the edges because we want to retain the information of the vertex of the previous step to avoid backtracking walks.

Note that, from the definition of P, one has that $P^T \mathbb{1} = \mathbb{1} \in \mathbb{R}^{\hat{E}}$. Additionally, we define the matrices $T_D, H_D \in \{0, 1\}^{V \times \hat{E}}$

$$H_D := \sum_{uv \in \hat{E}} e_v e_{uv}^T \quad \text{and} \quad T_D := \sum_{uv \in \hat{E}} e_u e_{uv}^T.$$

The matrix T_D is the *tails matrix*, so for each $xy \in \hat{E}$ and for each $u \in V$ we have that $(T_D)_{u,xy} = [u = x]$. Similarly, we have thar H_D is the *head matrix*, so for each $xy \in \hat{E}$ and for each $u \in V$ we have that $(H_D)_{u,xy} = [u = y]$. Note that

$$T_D^T \mathbb{1} = \mathbb{1} \in \mathbb{R}^{\hat{E}}, \tag{3.12}$$

and

$$H_D^T \mathbb{1} = \mathbb{1} \in \mathbb{R}^{\hat{E}}. \tag{3.13}$$

Also, note that

$$T_D \mathbb{1} = \deg_G, \tag{3.14}$$

and

$$H_D \mathbb{1} = \deg_G. \tag{3.15}$$

One has that for $l \geq 1$,

$$\tilde{P}_l = D_G^{-1} T_D P^l H_D^T, \tag{3.16}$$

which implies that

$$\tilde{P}_l \mathbb{1} = D_G^{-1} T_D P^l H_D^T \mathbb{1} = D_G^{-1} T_D P^l \mathbb{1} = D_G^{-1} T_D \mathbb{1} = D_G^{-1} D_G \mathbb{1} = \mathbb{1}. \tag{3.17}$$

Combining (3.15) and (3.16),

$$\mathbb{1}^T D_G \tilde{P}_l = \mathbb{1}^T H_D^T = \mathbb{1}^T D_G = \deg_G^T. \tag{3.18}$$

Lemma 53. ([Chung Graham, 2016, Lemma 8]) Let $G = (V, E)$ be a connected graph. Then

(i) for each integer $j \geq 0$

$$\tilde{P}_j^T \deg_G = \deg_G;$$

(ii) For each vertex x and any integer $j \geq 0$

$$\sum_{u \in V} \deg_G(u) \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \tilde{w}(p) = \deg_G(x);$$

(iii) For each vertex u and for any $l \geq 0$

$$\sum_{j=0}^{l} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \tilde{w}(p) = e_u^T (I + \tilde{P}_1 + \cdots + \tilde{P}_l)\mathbb{1} = l + 1.$$

Proof. We have that for any vertices $u, x \in V$ and for any integer $j \geq 0$

$$(\tilde{P}_j^T)_{xu} = e_u^T \tilde{P}_j e_x = \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \tilde{w}(p).$$

Let $x \in V$ and let $j \geq 0$ be an integer. Using (3.13), (3.16), (3.18), and (3.15), one has that

$$\mathbb{1}^T D_G \tilde{P}_j e_x = \mathbb{1}^T D_G (D_G^{-1} T_D P^j H_D^T) e_x = \mathbb{1}^T P^j H_D^T e_x = \mathbb{1}^T H_D^T e_x = \mathbb{1}^T D_G e_x = \deg_G(x),$$

whence $\tilde{P}_j^T \deg_G = \deg_G$ and item (i) is proved. Hence

$$\sum_{u \in V} \deg_G(u) \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \tilde{w}(p) = \sum_{u \in V} \deg_G(u) e_u^T \tilde{P}_j e_x = \deg_G^T \tilde{P}_j e_x = \mathbb{1}^T D_G \tilde{P}_j e_x = \deg_G(x),$$

and item (ii) is proved. Using (3.17), we have that

$$e_u^T (I + \tilde{P}_1 + \cdots + \tilde{P}_j)\mathbb{1} = (j+1) e_u^T \mathbb{1} = j + 1.$$

Also, we have that

$$\sum_{i=0}^{j} \sum_{u \in V} \sum_{p \in \mathcal{P}_{x,u}^{(i)}} \tilde{w}(p) = \sum_{i=0}^{j} \sum_{u \in V} e_x^T \tilde{P}_i e_u = \sum_{i=0}^{j} e_x^T \tilde{P}_i \mathbb{1} = e_u^T (I + \tilde{P}_1 + \cdots + \tilde{P}_j)\mathbb{1} = j + 1,$$

and item (iii) is proved. □

3.3 Main Result

Let $G = (V, E)$ be a graph. Throughout this section , for each $u \in V$, we consider the function $g_u : V \to \mathbb{R}_+$ defined as

$$g_u(x) := \left(e_u^T (I + \tilde{P}_1 + \cdots + \tilde{P}_\ell)(x)\right)^{1/2} = \left(\sum_{j=0}^{l} \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \tilde{w}(p)\right)^{1/2}, \tag{3.19}$$

for each $x \in V$, where $\ell := \lfloor \mathrm{diam}(G)/4 \rfloor$, and where $\tilde{P}$ and $\tilde{w}$ are defined in (3.10) and (3.11), respectively.

Lemma 54. ([Chung Graham, 2016, Claim A]) Let $G = (V, E)$ be a connected weak Ramanujan graph. Set $\ell := \lceil \mathrm{diam}(G)/4 \rceil$. Define the function $g_u : V \to \mathbb{R}_+$, for each $u \in V$, as in (3.19). Then

$$\sum_{u \in v} \deg_G(u) \sum_{x \in V} g_u^2(x) \deg_G(x) = (\ell + 1) \|\deg_G\|^2.$$

Proof. Note that

$$\sum_{u \in v} \deg_G(u) \sum_{x \in V} g_u^2(x) \deg_G(x) = \sum_{u \in V} \sum_{j=0}^{\ell} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \deg_G(u) \tilde{w}(p) \deg_G(x),$$

by the definition of the function g_u. Since $(\tilde{P}_j)_{u,x} = \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \tilde{w}(p)$ for each $u, x \in V$ and for each integer $j \geq 0$,

$$\begin{aligned}
\sum_{u \in V} \sum_{j=0}^{\ell} \sum_{x \in V} \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \deg_G(u) \tilde{w}(p) \deg_G(x) &= \sum_{u \in V} \sum_{x \in V} \deg_G(u) \deg_G(x) \sum_{j=0}^{\ell} \sum_{p \in \mathcal{P}_{u,x}^{(j)}} \tilde{w}(p) \\
&= \sum_{u \in V} \sum_{x \in V} \deg_G(u) \deg_G(x) \sum_{j=0}^{\ell} (\tilde{P}_j)_{u,x} \\
&= \sum_{j=0}^{\ell} \deg_G^T \tilde{P}_j \deg_G.
\end{aligned}$$

Using item (i) of Lemma 53,

$$\sum_{j=0}^{\ell} \deg_G^T \tilde{P}_j \deg_G = \sum_{j=0}^{\ell} \deg_G^T \deg_G = (\ell + 1) \|\deg_G\|^2. \qquad \square$$

Lemma 55. ([Chung Graham, 2016, Claim B]) Let $G = (V, E)$ be a connected weak Ramanujan graph. Set $\ell := \lfloor \mathrm{diam}(G)/4 \rfloor$. Define the function $g_u : V \to \mathbb{R}_+$, for each $u \in V$, as in (3.19). Then

$$\sum_{u \in V} \deg_G(u) \sum_{xy \in E} (g_u(x) - g_u(y))^2 \leq (\ell + 1 - \ell \sigma_G) \|\deg_G\|^2.$$

Proof. By the definition of the family of function g_u,

$$\sum_{xy \in E} (g_u(x) - g_u(y))^2 = \sum_{xy \in E} \left(\sqrt{\sum_{j=0}^{\ell} \sum_{p \in \mathcal{P}_{u,x}^{(j)}} w(p)} - \sqrt{\sum_{j=0}^{\ell} \sum_{p \in \mathcal{P}_{u,y}^{(j)}} \tilde{w}(p)} \right)^2.$$

[Chung Graham, 2016, page 14] claims that, by using the inequality

$$\left(\sqrt{\sum_i a_i} - \sqrt{\sum_i b_i} \right)^2 \leq \sum_i (\sqrt{a_i} - \sqrt{b_i})^2,$$

one can prove that

$$\sum_{xy\in E}(g_u(x)-g_u(y))^2 = \sum_{xy\in E}\left(\sqrt{\sum_{j=0}^{\ell}\sum_{p\in\mathcal{P}_{u,x}^{(j)}}\tilde{w}(p)}-\sqrt{\sum_{j=0}^{\ell}\sum_{p\in\mathcal{P}_{u,y}^{(j)}}\tilde{w}(p)}\right)^2$$

$$\leq \sum_{t\leq\ell-1}\sum_{r\in V}\sum_{\substack{p\in\mathcal{P}_{u,r}^{(t)}\\ p'=p\cup s\in\mathcal{P}_{u,s}^{(t+1)}}}\left(\sqrt{\tilde{w}(p)}-\sqrt{\tilde{w}(p')}\right)^2+\sum_{x\in V}\sum_{p\in\mathcal{P}_{u,x}^{(\ell)}}\tilde{w}(p)(\deg_G(x)-1).$$

Note that p' is equal to p concatenated with s, hence $w(p') = w(p)/\deg_G(r)-1$, where r is the last vertex of p. So

$$\sum_{t\leq\ell-1}\sum_{r\in V}\sum_{\substack{p\in\mathcal{P}_{u,r}^{(t)}\\ p'=p\cup s\in\mathcal{P}_{u,s}^{(t+1)}}}\left(\sqrt{\tilde{w}(p)}-\sqrt{\tilde{w}(p')}\right)^2\leq\sum_{t\leq\ell-1}\sum_{x\in V}\sum_{p\in\mathcal{P}_{u,x}^{(t)}}\left(\sqrt{\tilde{w}(p)}-\sqrt{\frac{\tilde{w}(p)}{\deg_G(x)-1}}\right)^2(\deg_G(x)-1)$$

Expanding the quadratic term,

$$\sum_{t\leq\ell-1}\sum_{r\in V}\sum_{\substack{p\in\mathcal{P}_{u,r}^{(t)}\\ p'=p\cup s\in\mathcal{P}_{u,s}^{(t+1)}}}\left(\sqrt{\tilde{w}(p)}-\sqrt{\tilde{w}(p')}\right)^2\leq\sum_{t\leq\ell-1}\sum_{x\in V}\sum_{p\in\mathcal{P}_{u,x}^{(t)}}\tilde{w}(p)\left(\deg_G(x)-2\sqrt{\deg_G(x)-1}\right).$$

Note that

$$\sum_{x\in V}\sum_{p\in\mathcal{P}_{u,x}^{(\ell)}}\tilde{w}(p)(\deg_G(x)-1)\leq\sum_{x\in V}\sum_{p\in\mathcal{P}_{u,x}^{(\ell)}}\tilde{w}(p)\deg_G(x).$$

Hence, using item (ii) of Lemma 53,

$$\sum_{u\in V}\deg_G(u)\sum_{x\in V}\sum_{p\in\mathcal{P}_{u,x}^{(\ell)}}\tilde{w}(p)\deg_G(x)=\sum_{x\in V}\deg_G(x)^2. \tag{3.20}$$

Also, using item (ii) of Lemma 53,

$$\begin{aligned}&\sum_{u\in V}\deg_G(u)\sum_{t\leq\ell-1}\sum_{x\in V}\sum_{p\in\mathcal{P}_{u,x}^{(t)}}\tilde{w}(p)\left(\deg_G(x)-2\sqrt{\deg_G(x)-1}\right)\\ &=\sum_{t\leq\ell-1}\sum_{x\in V}\deg_G(x)\left(\deg_G(x)-2\sqrt{\deg_G(x)-1}\right)\\ &=\sum_{t\leq\ell-1}\sum_{x\in V}\deg_G(x)^2(1-\sigma_G)\\ &=\ell(1-\sigma_G)\sum_{x\in V}\deg_G(x)^2.\end{aligned} \tag{3.21}$$

Hence, by (3.21) and (3.20),

$$\sum_{u\in V}\deg_G(u)\sum_{xy\in E}(g_u(x)-g_u(y))^2 \le \ell(1-\sigma_G)\sum_{x\in V}\deg_G(x)^2+\sum_{x\in V}\deg_G(x)^2$$
$$=(\ell+1+\ell\sigma_G)\sum_{x\in V}\deg_G(x)^2. \qquad \square$$

Lemma 56. ([Chung Graham, 2016, Claim C]) Let $G=(V,E)$ be a connected weak Ramanujan graph. Set $\ell := \lfloor \mathrm{diam}(G)/4 \rfloor$. Define the function $g_u : V \to \mathbb{R}_+$, for each $u \in V$, as in (3.19). Then there exist a vertex $\tilde{u} \in V$ such that

$$\frac{g_{\tilde{u}}^T L_G g_{\tilde{u}}}{g_{\tilde{u}}^T D_G g_{\tilde{u}}} \le 1-\sigma_G\left(1-\frac{1}{l+1}\right)$$

Proof. By Lemma 55,

$$\sum_{u\in V}\deg_G(u) g_u^T L_G g_u = \sum_{u\in V}\deg_G(u)\sum_{xy\in E}(g_u(x)-g_u(y))^2 \le (\ell+1-\ell\sigma_G)\|\deg_G\|^2.$$

Using Lemma 54,

$$(\ell+1-\ell\sigma_G)\|\deg_G\|^2 = (\ell+1-\ell\sigma_G)\left(1-\frac{1}{l+1}\right)\sum_{u\in V}\deg_G(u)\sum_{x\in V} g_u(x)^2\deg_G(x).$$

Note that

$$(\ell+1-\ell\sigma_G)\left(1-\frac{1}{\ell+1}\right) = \left(\frac{\ell+1}{\ell+1}-\frac{\ell\sigma_G}{\ell+1}\right) = \left(1-\frac{\ell\sigma_G}{\ell+1}\right),$$

and that

$$\sum_{u\in V}\deg_G(u)\sum_{x\in V} g_u(x)^2\deg_G(x) = \sum_{u\in V}\deg_G(u) g_u^T D_G g_u.$$

Hence,

$$\sum_{u\in V}\deg_G(u) g_u^T L_G g_u \le \left(1-\frac{\ell\sigma_G}{\ell+1}\right)\sum_{u\in V}\deg_G(u) g_u^T D_G g_u.$$

Which implies that

$$\sum_{u\in V}\deg_G(u)\left(\left(1-\frac{\ell\sigma_G}{\ell+1}\right)g_u^T D_G g_u - g_u^T L_G g_u\right)\ge 0.$$

So, there must exist a vertex $\tilde{u}$ such that

$$\left(1-\frac{\ell\sigma_G}{\ell+1}\right)g_{\tilde{u}}^T D_G g_{\tilde{u}} - g_{\tilde{u}}^T L_G g_{\tilde{u}} \ge 0,$$

otherwise the sum over all the vertices would be negative. Let $\tilde{u}\in V$ such that

$$\left(1-\frac{\ell\sigma_G}{\ell+1}\right)g_{\tilde{u}}^T D_G g_{\tilde{u}} - g_{\tilde{u}}^T L_G g_{\tilde{u}} \ge 0.$$

Hence,

$$\begin{aligned}
&\left(1-\frac{\ell\sigma_G}{\ell+1}\right)g_{\tilde{u}}^T D_G g_{\tilde{u}} - g_{\tilde{u}}^T L_G g_{\tilde{u}} \geq 0\\
\Rightarrow&\left(1-\frac{\ell\sigma_G}{\ell+1}\right)g_{\tilde{u}}^T D_G g_{\tilde{u}} \geq g_{\tilde{u}}^T L_G g_{\tilde{u}}\\
\Rightarrow& 1-\frac{\ell\sigma_G}{\ell+1} \geq \frac{g_{\tilde{u}}^T L_G g_{\tilde{u}}}{g_{\tilde{u}}^T D_G g_{\tilde{u}}}.
\end{aligned}$$

Note that

$$\sigma_G\left(1-\frac{1}{\ell+1}\right)=\sigma_G\left(\frac{\ell+1}{\ell+1}-\frac{1}{\ell+1}\right)=\sigma_G\frac{\ell}{\ell+1}.$$

Hence,

$$1-\sigma_G\left(1-\frac{1}{\ell+1}\right) \geq \frac{g_{\tilde{u}}^T L_G g_{\tilde{u}}}{g_{\tilde{u}}^T D_G g_{\tilde{u}}}.$$ □

Theorem 57. ([Chung Graham, 2016, Theorem 9]) Let $G=(V,E)$ be a connected graph such that $\delta(G)\geq 2$ and $G\neq K_V$. Suppose that $\sigma_G\leq 1/2$ and $\operatorname{diam}(G)(1.5)^{\operatorname{diam}(G)}\geq \sigma^{-1}$. Then

$$\lambda_2^{\uparrow}(\mathcal{L}_G)\leq 1-\sigma_G\left(1-\frac{5}{\operatorname{diam}(G)}\right).$$

Proof. We may assume that G is weak Ramanujan, otherwise we would have $\lambda_2^{\uparrow}(\mathcal{L}_G) \leq 1-\sigma_G$ and the result is trivial. Set $\ell := \lfloor \operatorname{diam}(G)/4 \rfloor$ and define the function $g_u : V \to \mathbb{R}_+$, for each $u\in V$, as in (3.19). By Lemma 56, there exists a vertex $\tilde{u}$ which satifies

$$\frac{g_{\tilde{u}}^T L_G g_{\tilde{u}}}{g_{\tilde{u}}^T D_G g_{\tilde{u}}} \leq 1-\sigma_G\left(1-\frac{1}{\ell+1}\right).$$

Set $g := g_{\tilde{u}}$. Define

$$\beta := \frac{\sum_{x\in V} g(x)\deg_G(x)}{\sum_{x\in V}\deg_G(x)} = \frac{g^T\deg_G}{\mathbb{1}^T\deg_G},$$

and define the function $h : V \to \mathbb{R}^V$ as

$$h := g-\beta\mathbb{1}.$$

Note that

$$\deg_G^T h = \deg_G^T g - \frac{g^T\deg_G}{\mathbb{1}^T\deg_G}\deg_G^T\mathbb{1} = \deg_G^T g - g^T\deg_G = 0,$$

whence $h\perp\deg_G$. Also, note that

$$h^T L_G h = (g-\beta\mathbb{1})^T L_G (g-\beta\mathbb{1}) = g^T L_G g - 2\beta\mathbb{1}^T L_G g + \beta^2\mathbb{1}^T L_G\mathbb{1} = g^T L_G g,$$

and

$$\begin{aligned}
h^T D_G h &= (g-\beta\mathbb{1})^T D_G (g-\beta\mathbb{1}) = g^T D_G g - 2\beta\mathbb{1}^T D_G g + \beta^2\mathbb{1}^T D_G \mathbb{1} \\
&= g^T D_G g - 2\frac{(g^T \deg_G)^2}{\mathrm{vol}_G(V)} + \beta^2 \mathrm{vol}_G(V) = g^T D_G g - 2\frac{(g^T \deg_G)^2}{\mathrm{vol}_G(V)}\frac{\mathrm{vol}_G(V)}{\mathrm{vol}_G(V)} + \beta^2 \mathrm{vol}_G(V) \\
&= g^T D_G g - 2\beta^2 \mathrm{vol}_G(V) + \beta^2 \mathrm{vol}_G(V) = g^T D_G g - \beta^2 \mathrm{vol}_G(V).
\end{aligned}$$

Using Lemma 45,

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \le \frac{h^T L_G h}{h^T D_G h} = \frac{g^T L_G g}{g^T D_G g - \beta^2 \mathrm{vol}_G(V)}. \tag{3.22}$$

Since $\mathrm{supp}(g) \subseteq B_{\tilde{u}}(\ell)$,

$$(g^T \deg_G)^2 = \left(\sum_{x\in V} g(x)\deg_G(x)\right)^2 = \left(\sum_{x\in B_{\tilde{u}}(\ell)} g(x)\deg_G(x)\right)^2.$$

Hence, by Cauchy-Schwarz inequality,

$$\begin{aligned}
\left(\sum_{x\in B_{\tilde{u}}(\ell)} g(x)\deg_G(x)\right)^2 &\le \sum_{x\in B_{\tilde{u}}(\ell)} \deg_G(x) \sum_{x\in B_{\tilde{u}}(\ell)} g^2(x)\deg_G(x) \\
&= \mathrm{vol}_G(B_{\tilde{u}}(\ell)) \sum_{x\in B_{\tilde{u}}(\ell)} g^2(x)\deg_G(x) \\
&= \mathrm{vol}_G(B_{\tilde{u}}(\ell)) g^T D_G g.
\end{aligned}$$

So,

$$\beta^2 = \frac{(g^T \deg_G)^2}{\mathrm{vol}_G(V)^2} \le \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)^2} g^T D_G g. \tag{3.23}$$

By combining (3.22) and (3.23),

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \le \frac{g^T L_G g}{g^T D_G g - \beta^2 \mathrm{vol}_G(V)} \le \frac{g^T L_G g}{g^T D_G g\left(1 - \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)}\right)}.$$

By assumption,

$$\frac{g^T L_G g}{g^T D_G g} \le 1 - \sigma_G\left(1 - \frac{1}{\ell+1}\right).$$

So,

$$\lambda_2^{\uparrow}(\mathcal{L}_G) \le \frac{g^T L_G g}{g^T D_G g\left(1 - \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)}\right)} \le \frac{1 - \sigma_G\left(1 - \frac{1}{\ell+1}\right)}{1 - \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)}}.$$

By Theorem 52 with $\varepsilon = \sigma_G/\mathrm{diam}(G)$,

$$\frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)} \le \varepsilon.$$

Note that $\mathrm{diam}(G) \geq (\ell+1)/\ell$, otherwise G would be the complete graph. Hence,

$$\frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)} \leq \sigma_G \frac{\ell}{\ell+1}.$$

So,

$$1 - \sigma_G(1 - \frac{1}{\ell+1}) + \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)} \leq 1.$$

One can check that

$$1 - \sigma_G(1 - \frac{1}{\ell+1}) + \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)} \leq 1,$$

implies that

$$\frac{1 - \sigma_G\left(1 - \frac{1}{\ell+1}\right)}{1 - \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)}} \leq 1 - \sigma_G(1 - \frac{1}{\ell+1}) + \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)}.$$

Hence,

$$\begin{aligned}
\lambda_2^{\uparrow}(\mathcal{L}_G) &\leq \frac{1 - \sigma_G\left(1 - \frac{1}{\ell+1}\right)}{1 - \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)}} \leq 1 - \sigma_G(1 - \frac{1}{\ell+1}) + \frac{\mathrm{vol}_G(B_{\tilde{u}}(\ell))}{\mathrm{vol}_G(V)} \\
&\leq 1 - \sigma_G(1 - \frac{1}{\ell+1}) + \varepsilon = 1 - \sigma_G(1 - \frac{1}{\ell+1}) + \frac{\sigma_G}{\mathrm{diam}(G)}.
\end{aligned}$$

Note that $\ell \geq \mathrm{diam}(G)/4 - 1$. Hence, one has $\ell + 1 \geq \mathrm{diam}(G)/4$. So,

$$\begin{aligned}
\lambda_2^{\uparrow}(\mathcal{L}_G) &\leq 1 - \sigma_G\left(1 - \frac{1}{\ell+1}\right) + \frac{\sigma_G}{\mathrm{diam}(G)} \\
&\leq 1 - \sigma_G\left(1 - \frac{4}{\mathrm{diam}(G)}\right) + \frac{\sigma_G}{\mathrm{diam}(G)} \\
&= 1 - \sigma_G\left(1 - \frac{5}{\mathrm{diam}(G)}\right).
\end{aligned}$$

□

Chapter 4

Conclusion

In this monograph, generalizations of expander graphs and Alon-Boppana-type bounds for each generalization were studied. The first generalization uses the notion of spectral sparsifiers of complete graphs, which are strongly related to the ratio of the largest eigenvalue of the Laplacian matrix to the second smallest eigenvalue of the Laplacian matrix. During the proof of the Alon Boppana bound in chapter 2, random non-backtracking walks were used to prove the existence of a vertex r that could produce a function f_r, which was then used in a Rayleigh quotient to bound the ratio of the largest eigenvalue of the Laplacian matrix and the second smallest eigenvalue of the Laplacian matrix.

The second generalization uses the normalized Laplacian matrix. During the proof of the Alon Boppana bound in chapter 3, the notion of weak Ramanujan graphs was used along with non backtracking walks. However, in this case, a matrix $\tilde{P}$, indexed by the edges of the graph, was employed to store the previous step of the walk, thereby avoiding backtracks.

The proofs of both Alon-Boppana bounds involve several interesting methods and concepts that may be useful in other contexts. Furthermore, expander graphs have already proven to be useful in their own right.

References

[Awoki 2020] Karina Suemi Awoki. “Árvores entrelaçadoras de polinômios e grafos de Ramanujan”. MA thesis. University of São Paulo, 2020.

[Batson *et al.* 2012] Joshua Batson, Daniel A. Spielman, and Nikhil Srivastava. “Twice-Ramanujan sparsifiers”. *SIAM J. Comput.* 41.6 (2012), pp. 1704–1721. issn: 0097-5397,1095-7111. doi: 10.1137/090772873. url: https://doi.org/10.1137/090772873 (cit. on pp. 11, 14).

[Chung Graham 2016] Fan Chung Graham. “A generalized alon-boppana bound and weak ramanujan graphs”. *Electron. J. Comb.* 23 (2016), p. 3. url: https://doi.org/10.37236/5933 (cit. on pp. 39, 40, 44–49, 51, 53, 54).

[Hoory *et al.* 2006] Shlomo Hoory, Nathan Linial, and Avi Wigderson. “Expander graphs and their applications”. *American Mathematical Society. Bulletin. New Series* 43.4 (2006), pp. 439–561. url: https://doi.org/10.1090/S0273-0979-06-01126-8 (cit. on pp. 8, 9).

[Marcus *et al.* 2015] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. “Interlacing families i: bipartite ramanujan graphs of all degrees”. *Annals of Mathematics. Second Series* 182.1 (2015), pp. 307–325. url: https://doi.org/10.4007/annals.2015.182.1.7 (cit. on p. 9).

[Nilli 1991] Alon Nilli. “On the second eigenvalue of a graph”. *Discrete Mathematics* 91.2 (1991), pp. 207–210. issn: 0012-365X. doi: https://doi.org/10.1016/0012-365X(91)90112-F. url: https://www.sciencedirect.com/science/article/pii/0012365X9190112F (cit. on pp. 1, 9, 13).

[Srivastava and Trevisan 2018] Nikhil Srivastava and Luca Trevisan. “An Alon-Boppana type bound for weighted graphs and lowerbounds for spectral sparsification”. In: *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms.* SIAM, Philadelphia, PA, 2018, pp. 1306–1315. isbn: 978-1-61197-503-1. doi: 10.1137/1.9781611975031.85. url: https://doi.org/10.1137/1.9781611975031.85 (cit. on pp. 19, 20, 22, 23, 25–27, 29, 32).